The Calderón projector and Dirichlet-Neumann operator for fibred cusp geometries

(Conference 'Analytic and Geometric Aspects of Spectral Theory', Casa Matemática in Oaxaca)

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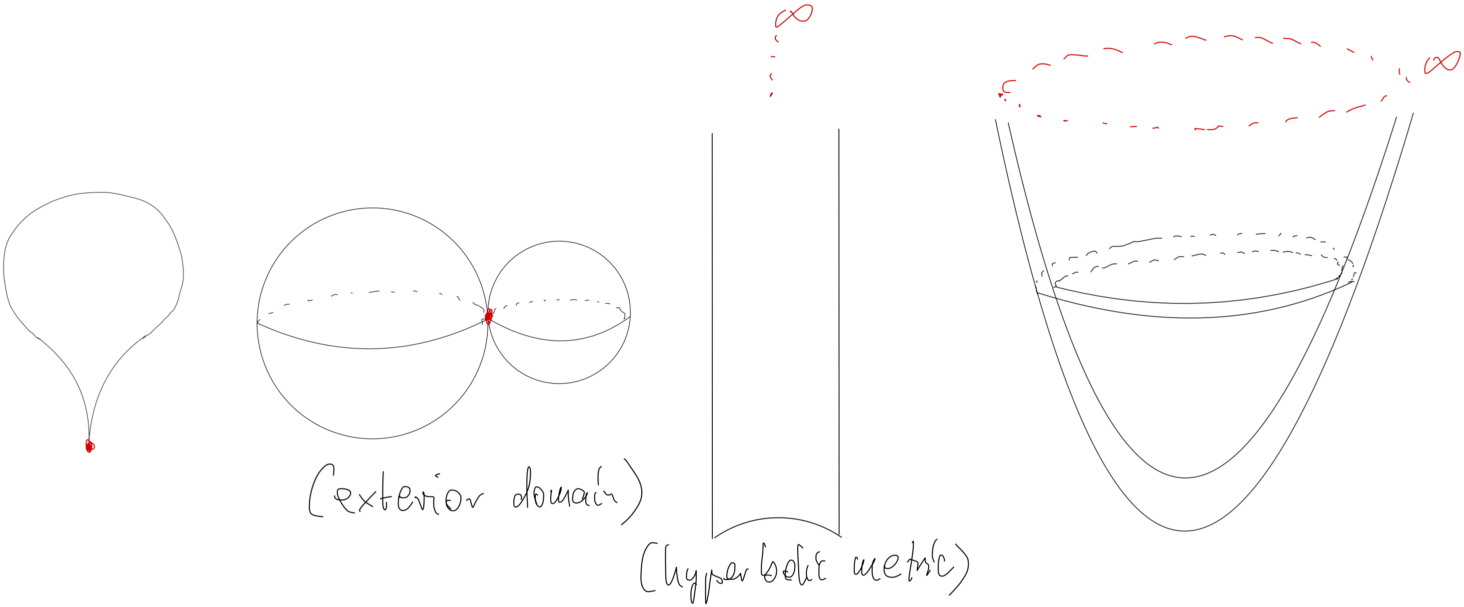
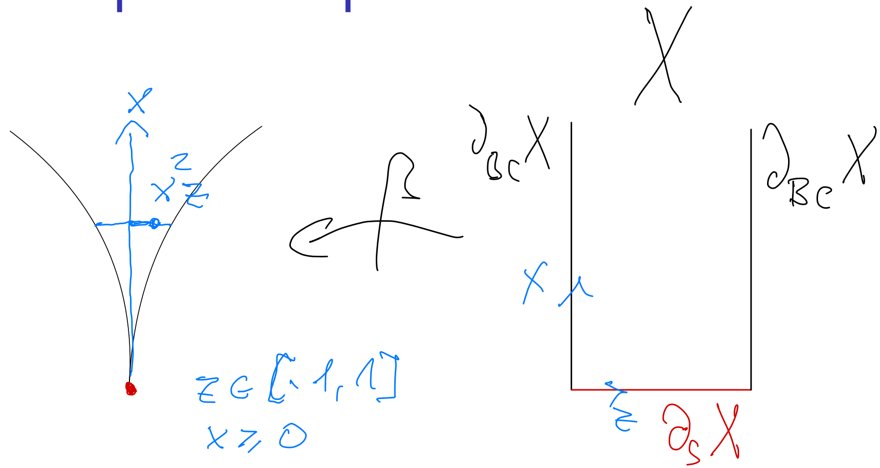
Plan of talk

1. Geometry: Fibred cusps
2. Analysis: Calderón projector and Dirichlet-Neumann operator
3. C and $\mathcal{N}$ are fibred cusp ΨDOs
4. What this means

How are these geometries related?



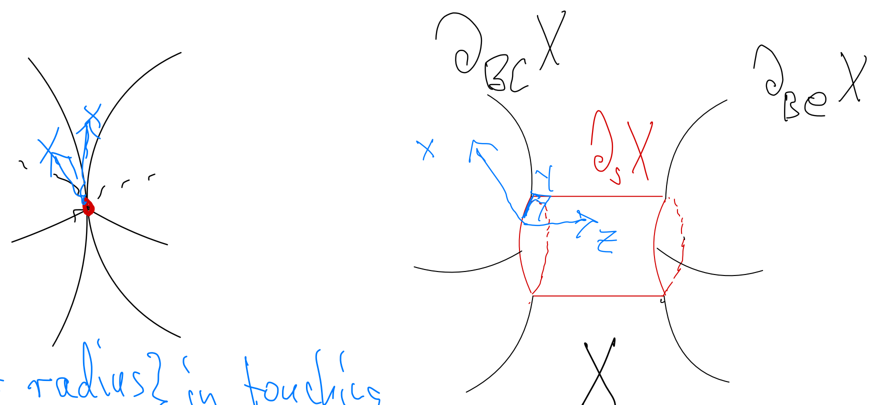
Incomplete cusp



Metric

$$g = (dx)^2 + (d(x^2 z))^2$$

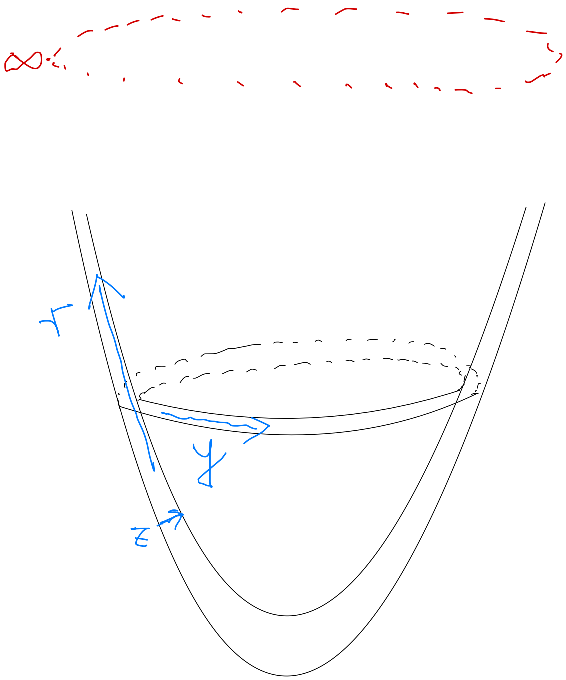
$$= (dx)^2 + (x^2 dz)^2 + \dots$$



x = radius } in touching
y = angle } plane

$$g = (dx)^2 + (x\,dy)^2 + (x^2 dz)^2 + \dots$$

Fat cone near infinity

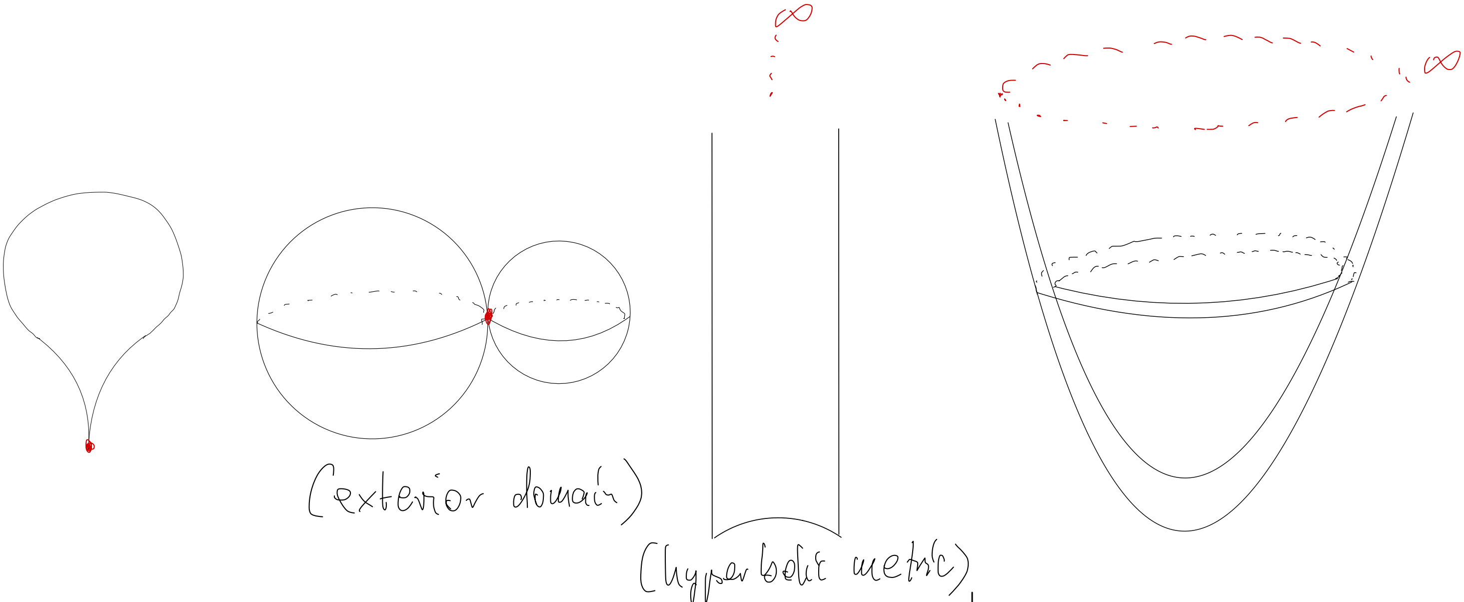


$x = \dfrac{1}{r}$, then infinity $= \{x = 0\}$

Metric:

$$
\begin{aligned}
&(dr)^2 + (rdy)^2 + (dz)^2 \\
&= \left(\frac{dx}{x^2}\right)^2 + \left(\frac{dy}{x}\right)^2 + (dz)^2
\end{aligned}
$$

How these geometries are related



$(dx)^2 + (xdy)^2 + (x^2dz)^2$ $= x^{-4} g_\varphi$	$\left(\frac{dx}{x}\right)^2 + (xdz)^2$ $= x^{-2} g_\varphi$	$\left(\frac{dx}{x^2}\right)^2 + \left(\frac{dy}{x}\right)^2 + (dz)^2$ $=: g_\varphi$

Note:

Always $y \in B =$ a closed manifold; $z \in F =$ a compact mfd with boundary

Fibred cusp (or φ-)manifolds, φ-metrics

Definition (Mazzeo/Melrose 1998)

1. φ-manifold: X = cpct mfd with boundary; fibration $\begin{array}{ccc} \underset{z}{F} & - & \partial X \\ & & \downarrow \varphi \\ & & \underset{y}{B} \end{array}$

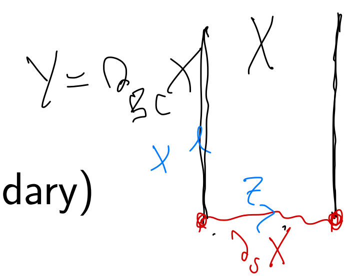
   $X \supset U \cong \underset{x}{[0, \varepsilon)} \times \underset{y,z}{\partial X}$

   $$\varphi\text{—metric: } g = \left(\frac{dx}{x^2}\right)^2 + \frac{\varphi^* g_B}{x^2} + g_F \approx \left(\frac{dx}{x^2}\right)^2 + \left(\frac{dy}{x}\right)^2 + (dz)^2 + \dots$$

2. φ-manifold with BC-boundary: F has non-empty boundary; then X has corners of codimension 2 and $\partial_{\mathrm{BC}} X$, $\partial_s X$
3. Cusp manifold (with BC-boundary): B = point, no y-variables

'Singularity' $\partial_s X$ always at $x = 0$

X has BC-boundary $\Rightarrow$
$\partial_{\mathrm{BC}} X$ is φ-manifold (without BC-boundary)



Cusp differential operators

In this talk: only cusp manifolds (no y-variables) – everything generalizes to φ

$$g = \left(\frac{dx}{x^2}\right)^2 + (dz)^2 + \cdots \Rightarrow \Delta_g = (x^2 D_x)^2 + D_z^2 + \ldots \qquad (D_x = \frac{1}{i}\partial_x)$$

Note: $g = x^{-2a}\tilde{g} \Rightarrow \Delta_g = x^{2a}\Delta_{\tilde{g}} + \ldots$

Definition

Example:

- Cusp differential operator:

$$P = \sum_{k+|\alpha|\leq m} a_{k\alpha}(x,z)(x^2 D_x)^k D_z^\alpha$$

$P = \Delta_g$

- Cusp principal symbol of P:

$$\sigma_c(P)(x,z;\tau,\zeta) = \sum_{k+|\alpha|=m} a_{k\alpha}(x,z)\tau^k\zeta^\alpha$$

$\tau^2 + \zeta^2$

- P is c-elliptic $:\Leftrightarrow$ $\sigma_c(P)$ invertible for $(\tau,\zeta) \neq 0$.

c-elliptic

- Normal family of P (a second symbol):

$$N(P)(\tau) = \sum_{k+|\alpha|\leq m} a_{k\alpha}(0,z)\tau^k D_z^\alpha \in \mathrm{Diff}^m(F)$$

$\tau^2 + D_z^2$

Calderón projector and Dirichlet-Neumann operator

Classical setting: X compact smooth manifold with boundary ($\partial_s X = \emptyset$)
Let $P \in \mathrm{Diff}^m(X)$ be elliptic. (acting on sections of vector bundles, if needed)

- Boundary data space:

$$\mathcal{B}_P := \{(u_{|\partial X}, \partial_\nu u_{|\partial X}, \dots, \partial_\nu^{m-1} u_{|\partial X}) : \ u \in C^\infty(X), Pu = 0\} \subset [C^\infty(\partial X)]^m$$

- Calderón projector: Any projection $C_P : [C^\infty(\partial X)]^m \to \mathcal{B}_P$
- Calderón 1963, Seeley 1966: $\exists\ C_P \in \Psi^*(\partial X; \mathbb{C}^m)$, formula for $\sigma(C_P)$.
- Dirichlet-Neumann operator $\mathcal{N} : C^\infty(\partial X) \to C^\infty(\partial X)$ for Δ:

$$\mathcal{N} f = (\partial_\nu u)_{|\partial X} \ \text{ if } u \text{ solves } \ \Delta u = 0, u_{|\partial X} = f$$

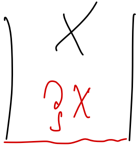
- Relation of $C_\Delta = (C_{ij})_{i,j\in\{0,1\}}$ of and $\mathcal{N}$:

$$\mathcal{N} = (I - C_{11})^{-1} C_{10}$$

- This implies

$$\mathcal{N} \in \Psi^1(\partial X)\,, \quad \sigma(\mathcal{N}) = |\xi|$$

Main theorems (Fritzsch/G./Schrohe)



Let X be a cusp manifold with BC-boundary.

Theorem (Calderón projector)

Let
$$P = x^{-ma}\tilde{P}\,, \quad \tilde{P} \in \mathrm{Diff}_c^m(X) \text{ c-elliptic}$$

Then P has a Calderón projector

$$C_P \in \Psi_c^*(\partial_{\mathrm{BC}}X)$$

$\sigma_c(C_P)$ given by the usual formula.
$N(C_P)(\tau) =$ Calderón projector for $N(P)(\tau)$.

Theorem (Dirichlet-Neumann operator)

Let $g = x^{2a}\tilde{g}$ for a cusp metric $\tilde{g}$ on X. Then the D-N operator satisfies

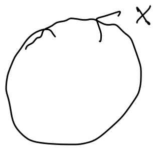
$$\mathcal{N} \in x^{-a}\Psi_c^1(\partial_{\mathrm{BC}}X)$$

$\sigma_c(\mathcal{N}) = |\xi|_g\,, \qquad \xi = (\tau, \zeta)$
$N(x^a\mathcal{N})(\tau) =$ the DN-operator of $\Delta_F + \tau^2 \approx -\partial_z^2 + \tau^2$.

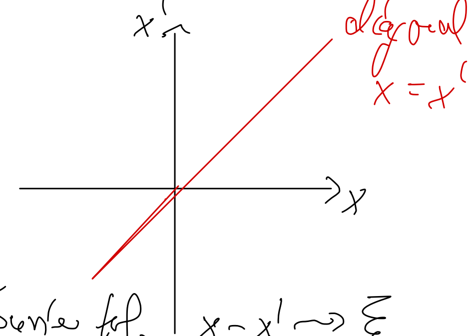
Note: Dirichlet problem $N(x^{-2a}\Delta)(\tau)$ is invertible $\forall\tau$.
($\hat{=}$ "full ellipticity")

What this means

classical: closed X



$$(Pu)(x) = \int_X K(x,x')\,u(x')\,dx', \quad K \in \mathcal{D}'(X \times X)$$



Diff(X): $P = \sum_k a_k(x)\,\partial_x^k$ $\qquad$ $K(x,x') = \sum_k a_k(x)\,\delta^{(k)}(x-x')$ $\rightsquigarrow$ Fourier tf. $x - x' \rightsquigarrow \xi$

$$\sum_k a_k(x)\,\xi^k$$

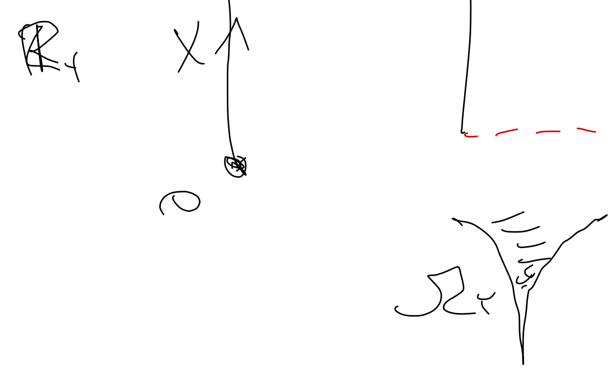
ξ generalize

ΨDO: $\qquad K(x,x')$:

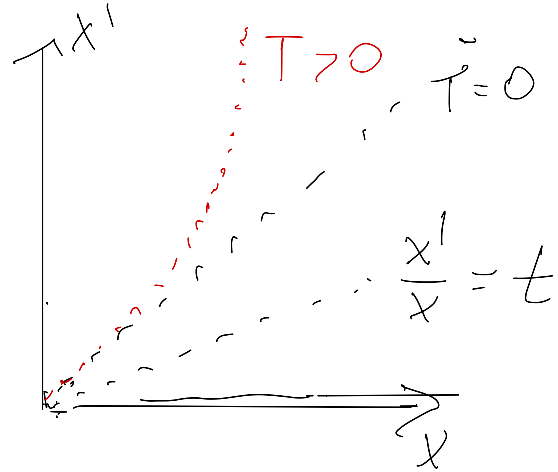
- smooth for $x \neq x'$
- conormal sing. at $x = x'$

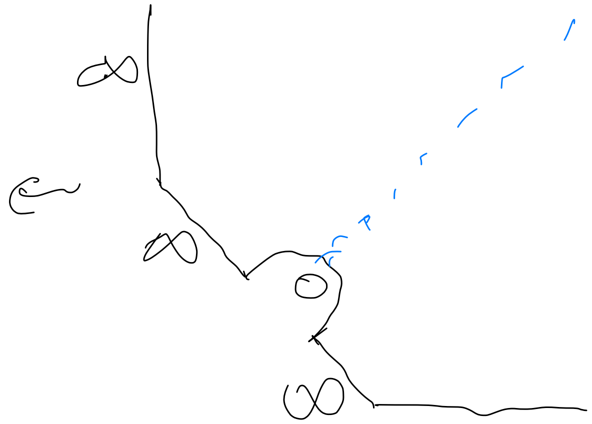
$\rightsquigarrow \sigma(x,\xi)$

$Y = \partial_{BC} X$



$Y \times_X Y$:

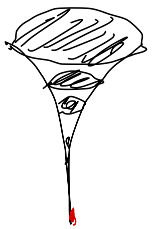




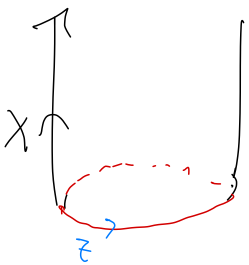
$k(x,x')$:

- $x \to 0$, $x' > \varepsilon$: $k = O(x^\infty)$
- $x \to 0$, $\left|\frac{x'}{x} - 1\right| > \varepsilon$: $k = O(x^\infty)$
- $x \to 0$, $\frac{x'}{x} - 1 = Tx$: k smooth in $x \geq 0$ $(T \neq 0)$
- con. sing. at $T = 0$.

Ω =



Y = $\partial_{BC} X$:



$Y \times_X Y$