Extremal Black Hole Corrections and the Weak Gravity Conjecture

Based on: 1905.06956 (**LA**, Alex Cole and Gary Shiu)
2011.05337 (**LA**, Alex Cole, Gregory Loges and Gary Shiu)
2111.xxxx (**LA**, appears tonight)

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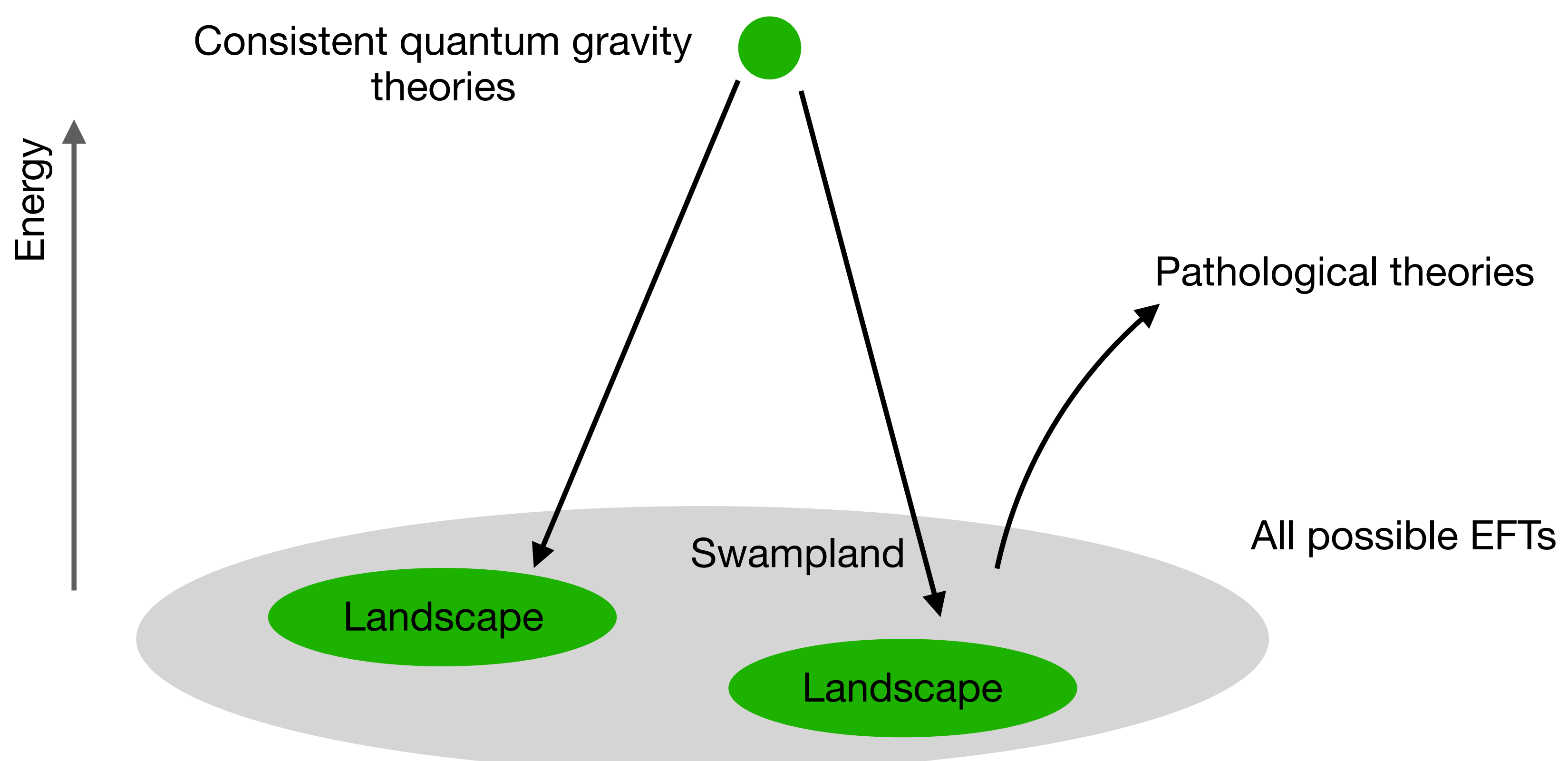
BIRS Workshop
Strings: Geometry and Symmetries for Phenomenology

November 8, 2021

Introduction

It has been appreciated or a long time that UV physics can constrain IR EFTs. This idea has been most sharply formulated in the context of the swampland. [Ooguri, Vafa '06]

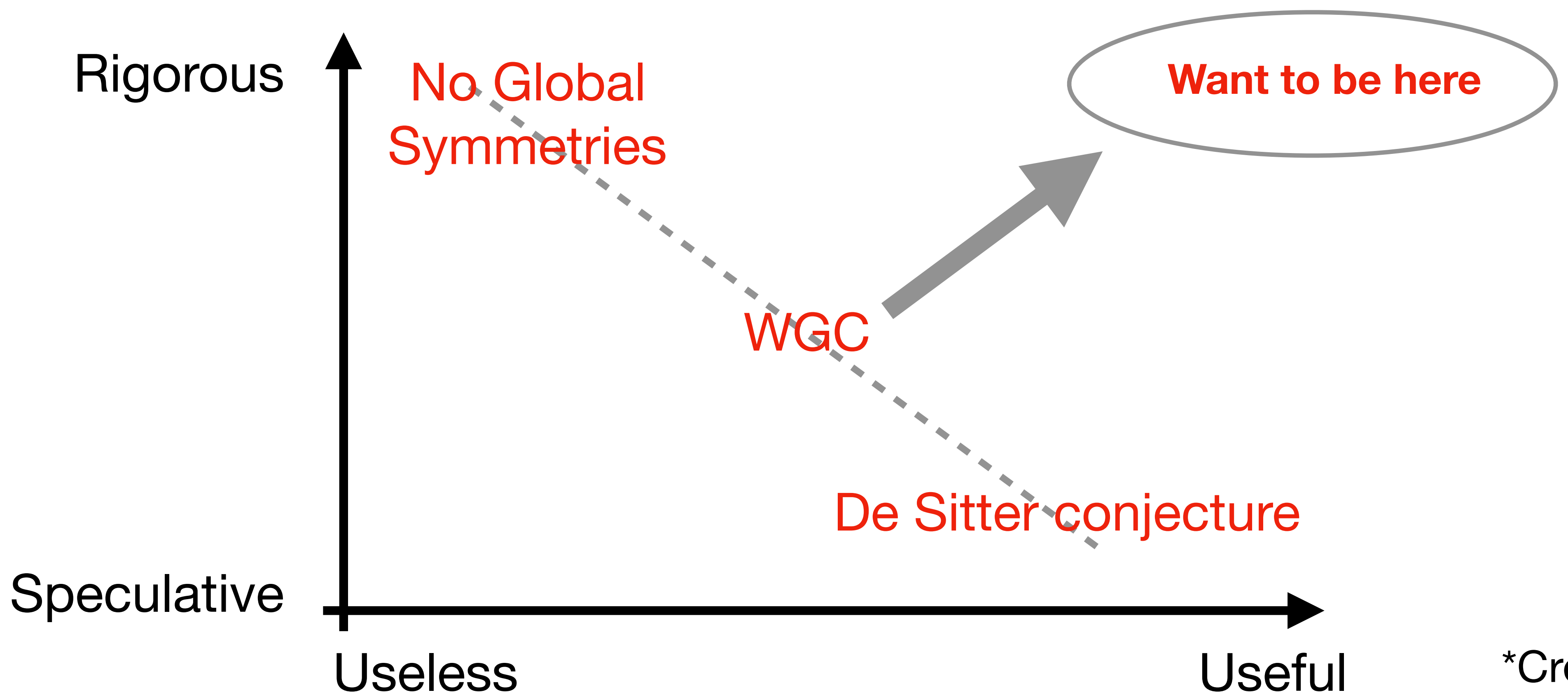


Swampland Conjectures

In order to distinguish the landscape from the swampland, various swampland conjectures have been proposed.

Their usefulness and rigorousness typically show an inverse correlation.

Ooguri-Hertzsprung-Russel Diagram*



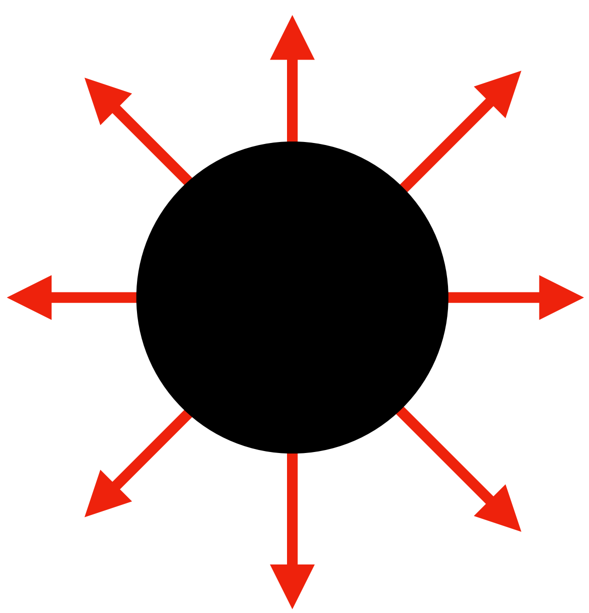
*Credit Eran Palti

Weak Gravity Conjecture

Today, I'll focus on the Weak Gravity Conjecture (WGC). This conjecture lies in a good place in the OHR diagram.

One can think of the WGC as a refinement of the statement that black holes do not carry global charges.

Can also consider gauge charges.



$$M \geq \sqrt{2} g Q M_p$$

However, taking the limit $g \to 0$ recovers the global symmetry.

Weak Gravity Conjecture

To prevent such a situation, a bound should exist that obstructs us from taking this limit: [Arkani-Hamed, Motl, Nicolis, Vafa '06]

WGC: "Any EFT with a U(1) gauge symmetry that arises as the low-energy limit of quantum gravity should contain a state that is *(super)extremal with respect to the black hole extremality bound.*"

For four-dimensional Reissner-Nordström black holes this becomes:

BH extremality:

$$M \geq \sqrt{2}Q$$

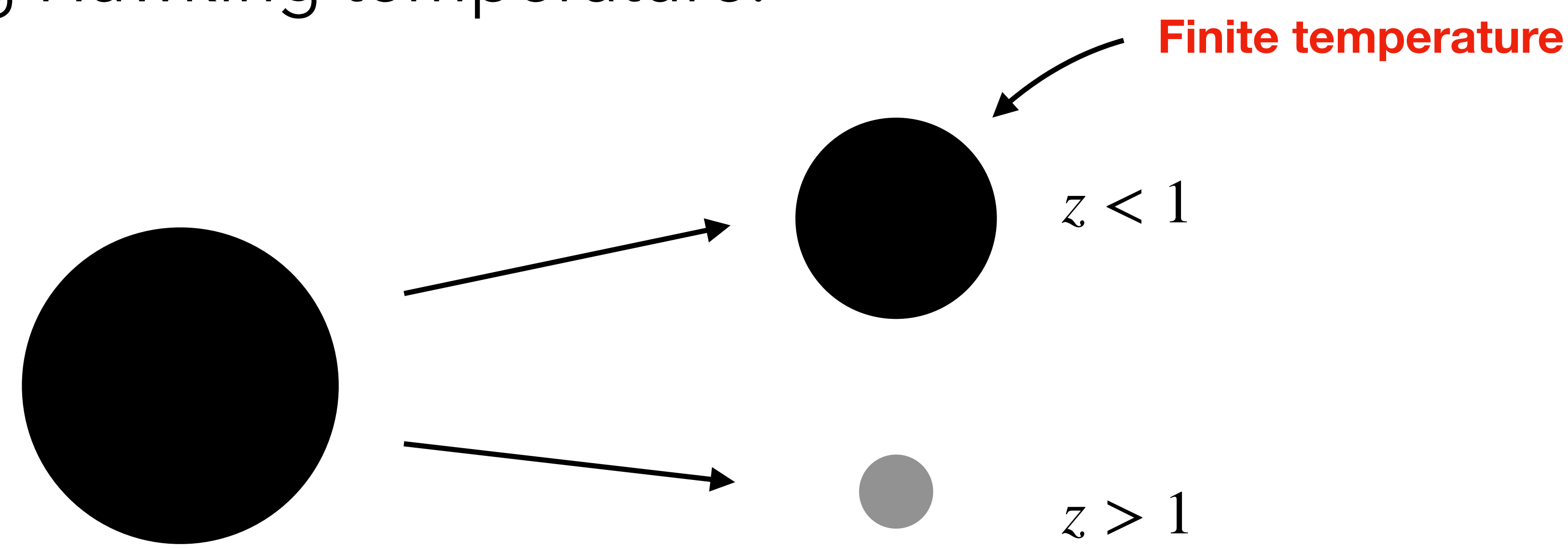
WGC:

$$m \leq \sqrt{2}q$$

Extremal Black Hole Decay

A corollary of the WGC is that it allows extremal black holes to decay despite a vanishing Hawking temperature.

$$z = \frac{\sqrt{2}Q}{M} = 1$$



Schwinger pair production

This prevents the possibility of forming a large tower of stable states not protected by any symmetry.

BPS states saturate the WGC bound, but non-supersymmetric states are expected to strictly satisfy it.

Evidence for the WGC

Most evidence for the WGC comes from string theory examples, but no (complete) proof has been found yet.

Most rigorous tests have been done in supersymmetric settings, but some non-supersymmetric results exist.

Heterotic String: $$M_s^2 = \frac{4}{\alpha'} N_R + Q_R^2 = \frac{4}{\alpha'}(N_L - 1) + Q_L^2$$

BPS Sates

$$N_R = 0 \quad \rightarrow \quad \frac{Q_R}{M_s} = 1$$

Non-BPS Sates

$$N_R = 0 \quad \rightarrow \quad \frac{Q_R}{M_s} > 1$$

Can we find more evidence away from the lamppost?

Outline

- **Mild Form of Weak Gravity Conjecture**
- Universal Relation for Extremality Corrections
- Applications to Higher-Derivative Corrected Black Holes
- Conclusions

Mild Form of the WGC

The WGC does not specify the mass of the state that should satisfy it.

It is therefore logically possible that black holes themselves play the role of WGC-satisfying states.

This is possible, because higher-derivative corrections modify the extremality bound. [Arkani-Hamed, Motl, Nicolis, Vafa '06] [Kats, Motl, Padi '06]

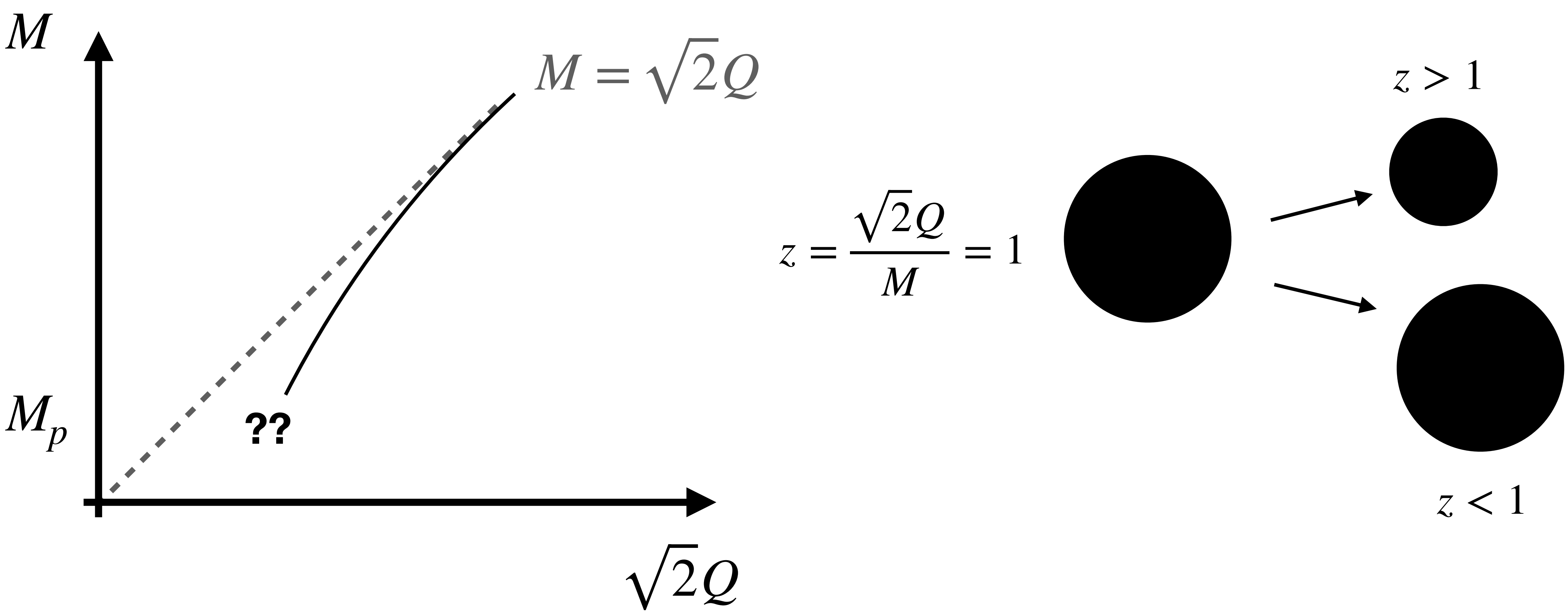
Leading Corrections: $$L = \frac{1}{2}R - \frac{1}{4}F_{ab}F^{ab} + \frac{a_1}{4}(F_{ab}F^{ab})^2 + \frac{a_2}{2}F_{ab}F_{cd}W^{abcd}$$

Extremality Bound: $$\frac{\sqrt{2}Q}{M} \leq 1 - \frac{32\pi^2(2a_1 - a_2)}{Q^2}$$

The WGC requires: $2a_1 - a_2 \geq 0$

Extremal Black Hole Decay

Again, this allows extremal black holes to decay. This time however, they do so by emitting a smaller extremal black hole.



The first part of the curve is monotonic, until the higher-derivative expansion breaks down and we can't compute.

Arguments for Positivity

Again, in string theory examples the combination $2a_1 - a_2$ turns out to be positive. [Kats, Motl, Padi '06]

In addition, unitarity and causality can be used to argue that $a_1 \geq 0$, but there is no-general model-independent bound on a_2. [Hamada, Noumi, Shiu '18] [Arkani-Hamed, Huang, Liu, Remmen '21]

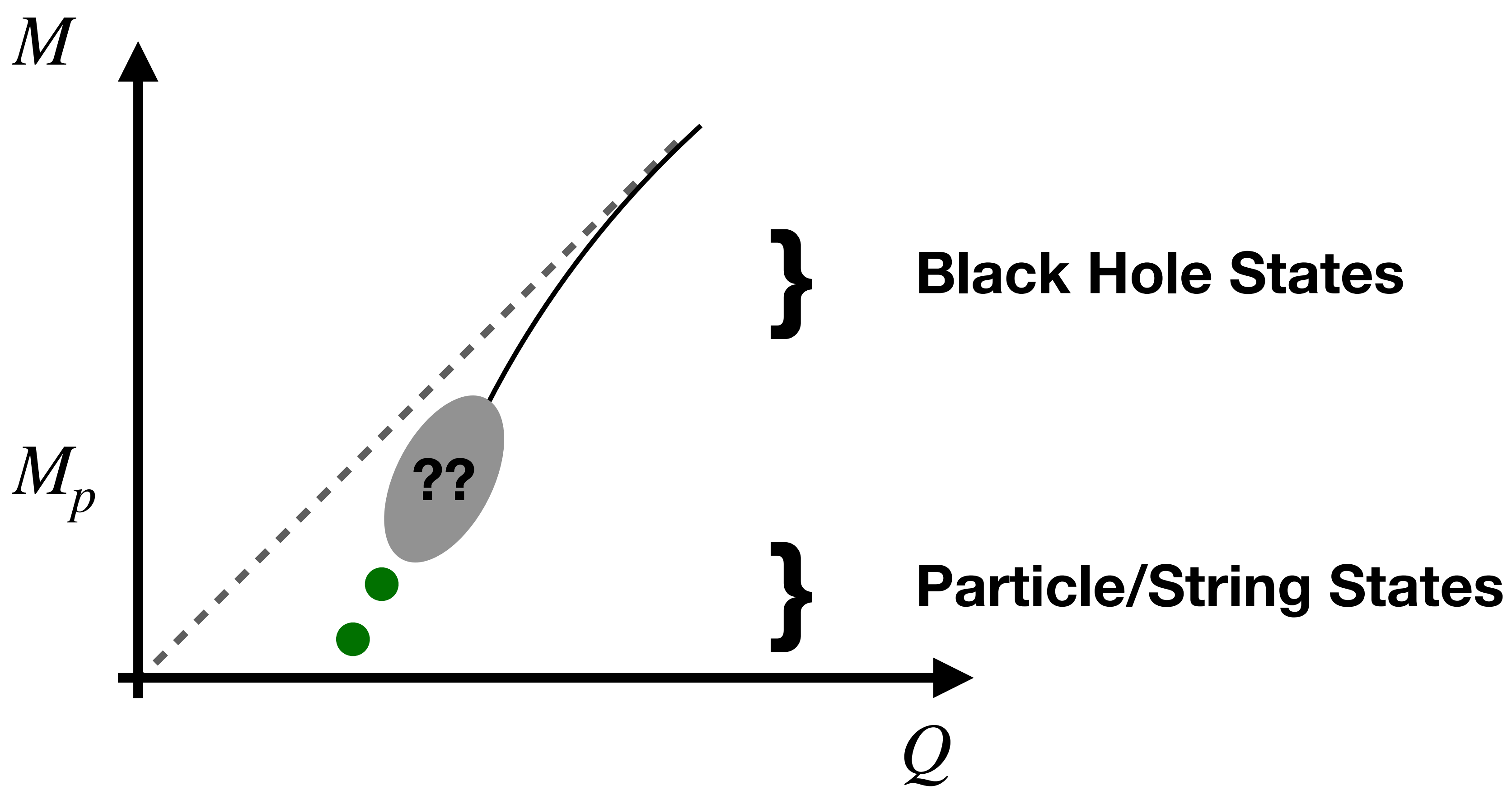
Instead, additional assumptions about the UV-completion are needed. When the gravitational effect of FFW is subdominant, WGC follows.

What structure of quantum gravity do we need to proof the (mild form) of the WGC?

Monotonicity

We can make headway by connecting the particle with the mild form of the WGC.

Like to show that the charge/mass ratio curve is monotonic.



For BPS states this is relatively straightforward. Non-supersymmetric states?

Worldsheet CFT

Consider the left-moving sector of a 2d CFT partition function with a level k $U(1)$current:

$$q = e^{2\pi i\tau}$$

$$y = e^{2\pi i\mu}$$

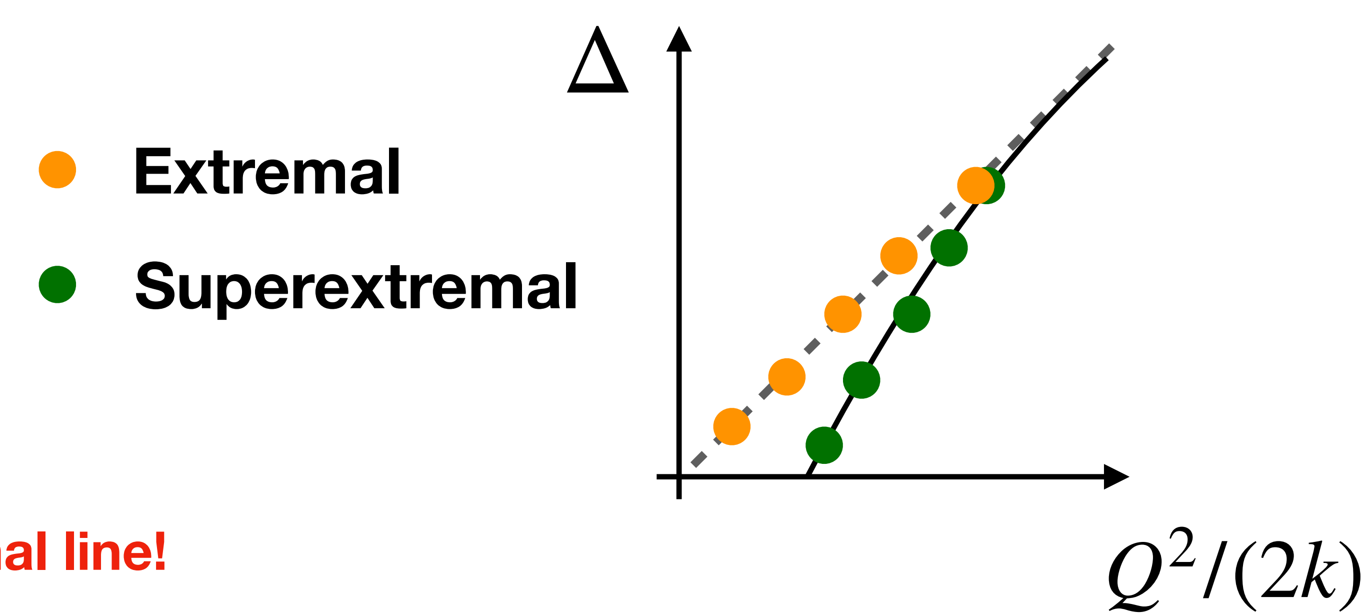
$$\Delta = L_0 - \frac{c}{24}$$

$$Z_L(\tau;\mu) = \mathrm{Tr}(q^{\Delta}y^{Q})$$

Modular invariance then allows us to apply the following transformation to generate a tower of states. [Montero, Shiu, Soler '16][Heidenreich, Reece, Rudelius '16][LA, Cole, Shiu '19]

$$L_0 \to L_0 + Q + k$$

$$Q \to Q + k$$



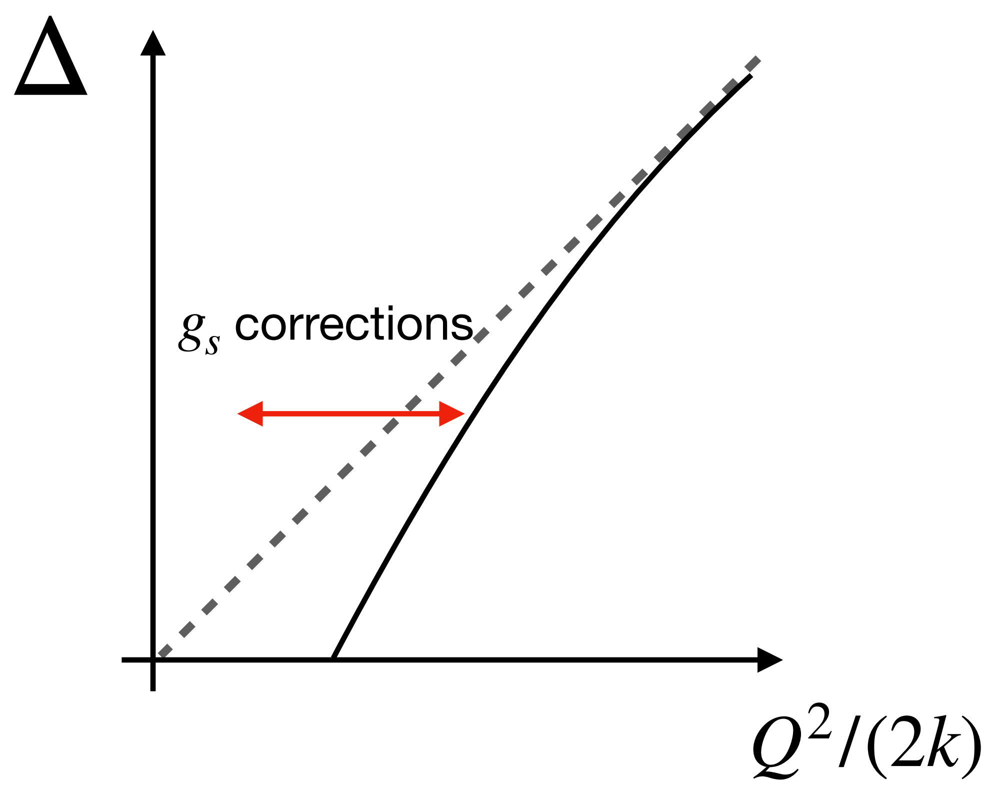
Superextremal states don't cross the extremal line!

Turning on Gravity

If we turn on g_s, states with $\Delta \geq \mathcal{O}(c)$ are expected to form black holes.

Can we use this to argue that the charge/mass curve is monotonic?

In general, no! g_s corrections modify the mass, possibly ruining the shape of the curve.

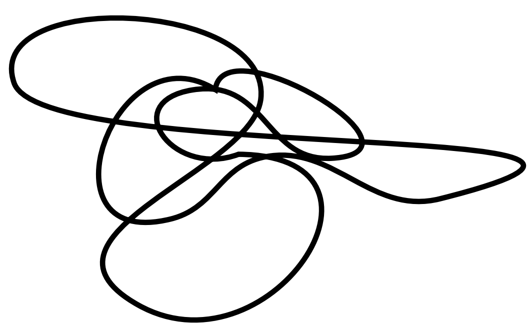


Can we control these corrections in non-supersymmetric cases?

Black Holes as Strings

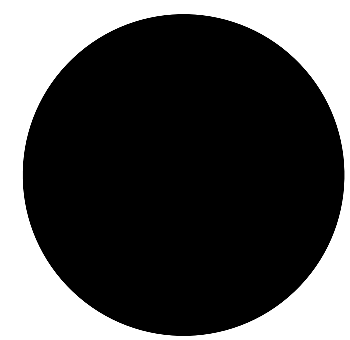
Consider a highly excited ($N \gg 1$) weakly coupled ($g_s \ll 1$) string, and adiabatically turn up the string coupling. [Horowitz, Polchinski '96]

$$G_N = g_s^2 l_s^2$$



$$\xrightarrow[g_s^2 \sim 1/(Ml_s)]{r_s \sim l_s}$$

$$r_s = 2G_N M$$



String Entropy: $S_s \sim \sqrt{N}$

String Mass: $M_s \sim \sqrt{N}/l_s$

Black Hole Entropy: $S_{bh} \sim G_N M^2$

At transition point: $S_s = \mathcal{O}(1) S_{bh}$ $\longleftarrow$ **Mass correction only induces $\mathcal{O}(1)$ factor**

Small Black Holes

There exists a class of (two-charge) small black holes, whose classical area is zero. However, it is expected that higher-derivative corrections stretch the horizon. [Dabholkar '04] [Sen'05]

These black holes enjoy BTZ near-horizon geometries whose entropy is given by Cardy's formula:

$$S = 2\pi\left(\sqrt{\frac{c_L}{6}N_L} + \sqrt{\frac{c_R}{6}N_R}\right)$$

For two-charge black holes in heterotic string theory, the central charges are fixed by anomalies. This yields α'-exact results: [Kraus, Larsen '05]

$$c_L = 24 \qquad\longleftarrow\qquad$$
$$c_R = 12$$

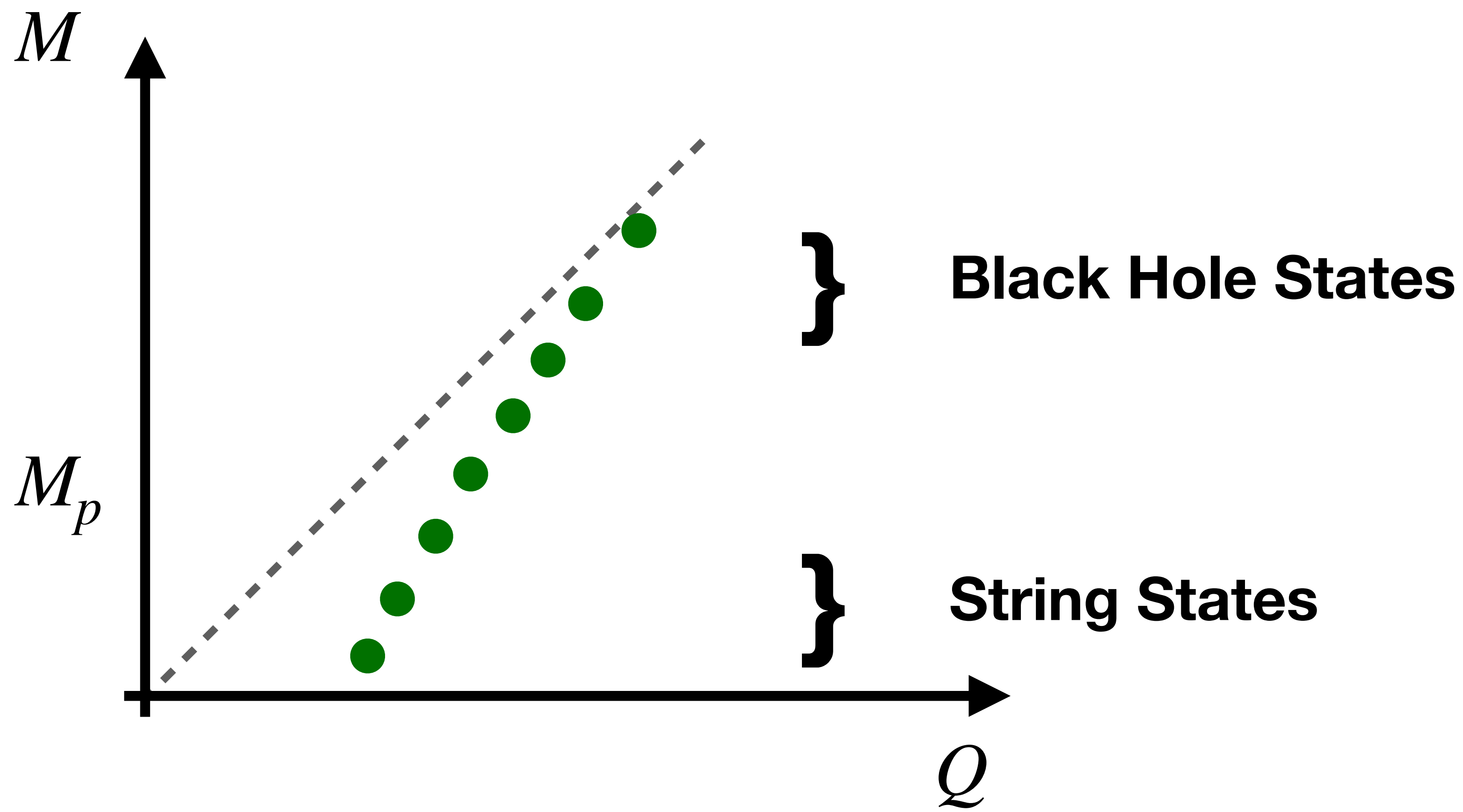
Coincides exactly with the entropy of a free heterotic string!

Exact Entropy Matching

The exact entropy matching implies that the mass of a heterotic string is not modified, but protected by anomalies.

This holds even for non-supersymmetric and non-extremal black holes! Thus, the charge/mass ratio is not corrected during the transition.

This implies that the curve is monotonic. [LA, Cole, Shiu '19]



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- Mild Form of Weak Gravity Conjecture
- **Universal Relation for Extremality Corrections**
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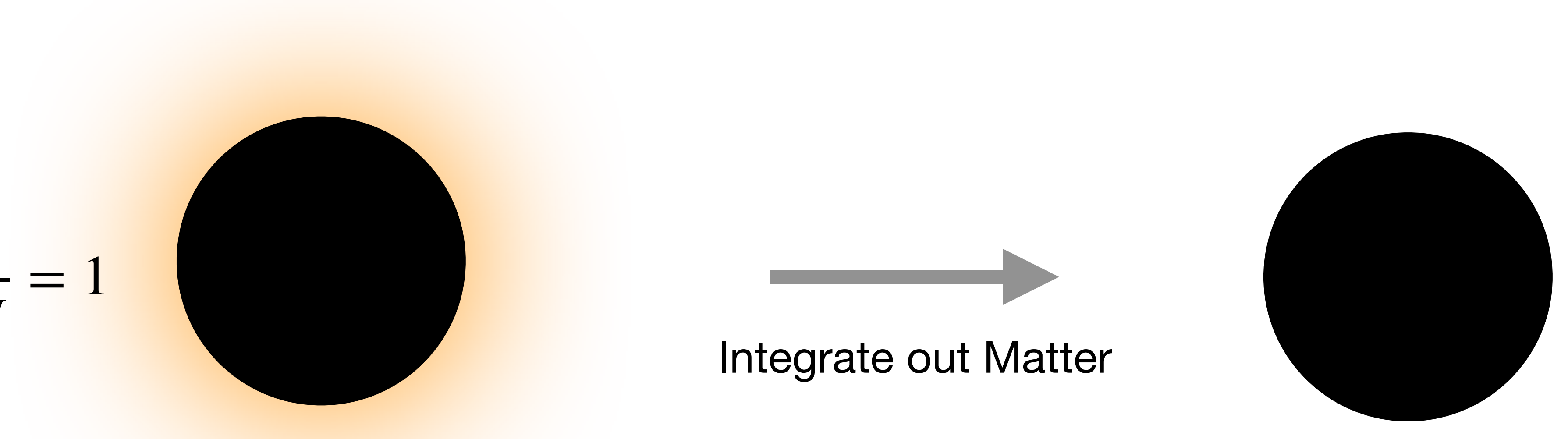
Universal Corrections

To test the mild form of the WGC for more general black holes, it is advantageous to have a more model-independent formulation.

$$\frac{Q}{M} = 1$$



$$\frac{Q}{M} = 1 + f(L_{\text{matter}})$$

$$L = L_{\text{GR}} + L_{\text{matter}}$$

In particular, what are the general conditions on perturbations to black holes that we should impose to satisfy the WGC?

We'll derive a universal expression that holds for Kerr-Newman black holes.

Iyer-Wald Formalism

Our starting point is a Lagrangian d-form $L(\phi)$ depending on some set of matter fields ϕ. [Wald '93] [Iyer, Wald '94]

$$\delta L(\phi) = E(\phi) + \mathrm{d}\theta(\phi, \delta\phi)$$

EOM   Boundary Terms

If the action has a symmetry generated by a vector field ξ we can define a Noether current.

$$J_\xi(\phi) = C_\xi + \mathrm{d}Q_\xi = \theta(\phi; \mathscr{L}_\xi\phi) - \iota_\xi L(\phi)$$

Constraints
=0 on EOM   Noether charge

Performing variations and integrating we find:

$$\int_{\partial\Sigma} \Big(\delta Q_\xi - \iota_\xi \theta(\phi, \delta\phi) \Big) = - \int_\Sigma \delta C_\xi$$

Einstein-Maxwell Theory

[Wald, Sorce '17][LA '21]

Evaluating this relation for four-dimensional Einstein-Maxwell theory, we see the use of it:

$$L = \frac{1}{2}(R - 2\Lambda) - \frac{1}{4}F_{ab}F^{ab}$$

Constraints:

$$C^g_{abc} = \epsilon_{abce}\xi^d (T^g)_d{}^e$$

$$C^A_{abc} = \epsilon_{abce}(\xi^d A_d) j^e$$

Charges:

$$Q^g_{ab} = \frac{\epsilon_{abcd}}{2}\nabla^c \xi^d$$

$$Q^A_{ab} = -\frac{\epsilon_{abcd}}{2}F^{cd}A_e\xi^e$$

The Kerr-Newman metric has a Killing vector: $K^a = t^a + \Omega\varphi^a$. Using this, we find:

$$\delta M - \Omega\delta J - \Phi\delta Q - \kappa\delta A_B = \int_\Sigma \mathrm{d}^3x\sqrt{h}\left[\delta T^g_{ab} + F_{ac}\delta F_b{}^c\right]K^a n^b$$

Black Hole Quantities **Perturbation**

Extremal Limits

[LA '21]

It will be useful to write the right-hand side as an "effective stress tensor": [LA '21]

$$\delta T^{\mathrm{eff}}_{\mathrm{ab}} := \delta(T^{g})_{ab} + F_{ac}\delta F_{b}^{\ c}$$

When $\delta T^{\mathrm{eff}}_{ab} = 0$, we find that extremal black holes obey:

$$T = 0 \qquad \longleftrightarrow \qquad \delta M - \Omega\delta J - \Phi\delta Q = 0$$

Thus, the sign of the integral over the effective stress tensor determines the correction to the extremality bound and entropy:

$$T \to 0: \qquad \delta M - \Omega\delta J - \Phi\delta Q = \int_{\Sigma} \mathrm{d}^3x\sqrt{h}T^{\mathrm{eff}}_{ab}K^a n^b$$

$$\delta M - \Omega\delta J - \Phi\delta Q \to 0: \qquad T\delta S_{\mathrm{BH}} = -\int_{\Sigma} \mathrm{d}^3x\sqrt{h}T^{\mathrm{eff}}_{ab}K^a n^b$$

Different Ensembles

[LA '21]

The WGC is typically evaluated in a canonical ensemble (fixed T, Q, J).

$$\lim_{T\to 0} (\delta M)_{T,Q,J} = \int_\Sigma \mathrm{d}^3x\sqrt{h}T^{\text{eff}}_{ab}K^a n^b$$

≤ 0 **when WGC is satisfied**

In a microcanonical ensemble (fixed M, Q, J) this is related to the entropy correction.

$$(T\delta S_{\text{BH}})_{M,Q,J} = -\int_\Sigma \mathrm{d}^3x\sqrt{h}T^{\text{eff}}_{ab}K^a n^b$$

This implies an entropy/extremality relation:

$$\lim_{T\to 0} (\delta M)_{T,J,Q} = -\lim_{M\to \Omega J+\Phi Q} \left(T\delta S_{\text{BH}}\right)_{M,J,Q}$$

Entropy/Extremality Relation

Similar relations have appeared before by studying corrected metrics and using Euclidean thermodynamic methods. [Cheung, Liu, Remmen '18] [Hamada, Noumi, Shiu '18] [Goon, Penco '19]

$$\lim_{T\to 0} (\delta M)_{T,J,Q} \leq 0 \qquad \longleftrightarrow \qquad \lim_{M\to \Omega J + \Phi Q} \left(T\delta S_{\mathrm{BH}}\right)_{M,J,Q} \geq 0$$

Importantly, our relation involves the Bekenstein-Hawking entropy and not the Wald entropy.

In particular, this implies that topological terms will not contribute.

They appear has boundary terms, contributing to the Wald entropy, but do not correct the area / extremality bound.

What condition should we impose to decrease the mass?

Energy Conditions

Remember the definition of the effective stress tensor:

$$\delta T^{\text{eff}}_{\text{ab}} := \delta(T^g)_{ab} + F_{ac}\delta F_b{}^c$$

It is related to the "usual" stress tensor as follows.

$$\delta T_{ab} = \delta T^{\text{eff}}_{ab} + \left(F_{ac}\delta F_{bc} - \frac{1}{2} g_{ab} F_{cd}\delta F^{cd} \right)$$

If (…) vanishes, extremality/entropy corrections are determined by the stress tensor.

WGC: $$\int_\Sigma \mathrm{d}^3x\sqrt{h}T_{ab}K^a n^b \leq 0$$

Can this be related to energy conditions?

Energy Conditions

Because n^a is the unit normal to Σ, by definition it is a timelike vector.

$$K = t^a + \Omega\varphi^a$$

K^a is timelike when $\Omega = 0$, otherwise character is metric dependent. Can always find a point where this is the case.

For n^a, K^a timelike vector, a necessary condition for the WGC is:

Mass Decrease: $\delta T_{ab}K^a n^b \leq 0$ $\longleftarrow$ Violation of the Dominant Energy Condition

In general, this condition is not sufficient.

Sufficient Condition

We still need to perform the integral over Σ, in principle a negative contribution can be overwhelmed by positive contributions along the integral

Violation of the DEC is sufficient when either:

1. An integrated condition on the stress tensor holds.

2. The DEC is violated along the entirety of Σ.

Option 1. is unlikely.

Can check what happens in specific cases.

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Charged Black Holes in AdS

[LA '21]

[Cremonini, Jones, Liu, McPeak '19]

We now consider the following four-derivative corrections in AdS_4:

$$L = \frac{1}{2}R + \frac{3}{\ell^2} - \frac{1}{4}F_{ab}F^{ab} + \frac{a_1}{4}(F_{ab}F^{ab})^2 + \frac{a_2}{2}F_{ab}F_{cd}W^{abcd}$$

Metric: $\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{k,2}^2$

$$K^a = t^a$$

timelike

$$f(r) = k - \frac{M}{4\pi} + \frac{Q^2}{32\pi^2 r^2} + \frac{r^2}{\ell^2}$$

$$A = -\frac{Q}{4\pi r}\mathrm{d}t\ .$$

Performing variations, we obtain δT^g_{ab} and j^a. This results in somewhat lengthy expressions.

Charged Black Holes in AdS

The effective stress tensor reduces to the usual stress tensor when contracted with K^a.

$$\delta T^{\text{eff}}_{ab} K^a = \delta T_{ab} K^a$$

Are energy conditions sufficient to constrain Wilson coefficients? For spherical black holes and black branes we find:

$k = (0,1)$:

DEC: $\Omega^{-1}_{k,2}\, \delta T_{ab} n^a K^b \Big|_{a_2=0} \sim -\, g(k; r, r_+)^2\, a_1$

NEC: $\Omega^{-1}_{k,2}\, \delta T_{ab} N^a N^b \sim -\, h(k; r, r_+)^2\, a_2$

Violation of DEC: $a_1 \geq 0$ ⟵ **Also follows from unitarity**

NEC: $a_2 \leq 0$

Extremal Corrections

Performing the integral, we obtain the following corrections to extremality/entropy:

$$\frac{5\ell^4}{12r_+^3\Omega_{k,2}}\int_\Sigma \mathrm{d}^3x\sqrt{h}\delta T_{ab}K^a n^b = -\frac{\left(k\ell^2+3r_+^2\right)\left(2a_1\left(k\ell^2+3r_+^2\right)-a_2\left(k\ell^2-2r_+^2\right)\right)}{6r_+^4}$$

For small black holes ($r_+/\ell \ll 1$):

$$\int_\Sigma \mathrm{d}^3x\sqrt{h}\delta T_{ab}K^a n^b \sim -(2a_1 - a_2)$$

For large black holes ($r_+/\ell \gg 1$) and black branes:

$$\int_\Sigma \mathrm{d}^3x\sqrt{h}\delta T_{ab}K^a n^b \sim -(3a_1 + a_2)$$

For small black holes, violation of DEC + NEC leads to the WGC.

Kerr Black Holes

[LA '21]

[Reall, Santos '19]

For Kerr black holes, there is just a single leading higher-derivative term:

$$L = \frac{1}{2}\left(R + \alpha R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab}\right)$$

However, due dependence on the azimuthal angle $\delta T_{ab} k^a l^a$ has no definite sign.

Performing the integral we find:

$$\int_\Sigma \mathrm{d}^3x \sqrt{h}\, \delta T_{ab} K^a n^b = \frac{4\pi\alpha}{35 r_+^3 (r_- + r_+)^3} \left(-37 r_-^3 + 357 r_-^2 r_+ - 455 r_- r_+^2 + 175 r_+^3\right) \ .$$

Positive definite!

The mass is corrected negatively when $\alpha < 0$, but there is no energy condition leading to this result.

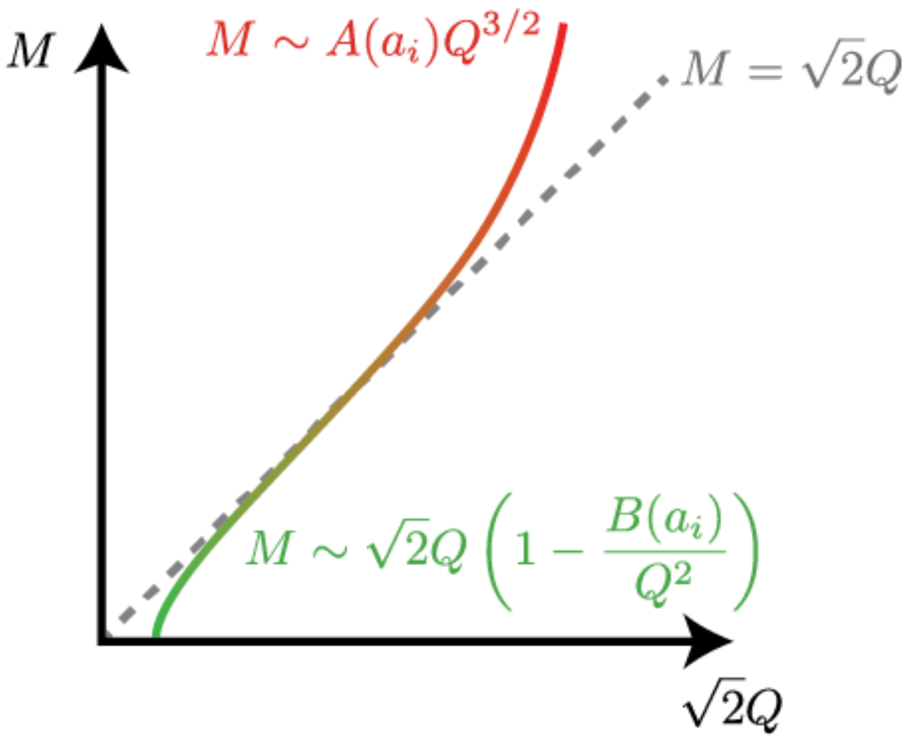
Mild Convexity in AdS?

For large BHs in AdS, it is not completely clear what the correct formulation of the WGC is.

Recently, in the context of AdS/CFT a Convex Charge Conjecture (CCC) was put forward: [Aharony, Palti '21]

CCC: $$\Delta(n_1 + n_2) \geq \Delta(n_1) + \Delta(n_2)$$

Could a mild form of this conjecture be true?



Large AdS BHs are always convex irrespective of the sign of Wilson coefficients

Conclusions

The mild form of the WGC is an interesting conjecture in the OHR diagram.

We investigated what ingredients of quantum gravity are necessary to satisfy it: in a particular example modular invariance was crucial.

Using the Iyer-Wald formalism, we formulated the mild WGC as a condition on the stress tensor.

A necessary (but not sufficient) condition is: violation of the DEC.

We hope this formalism will be useful in further sharpening how much UV information is needed to prove the WGC.

Thank you!

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