Power Concavity and Dirichlet heat flow

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Joint work with

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§1. Power concavity

$\Omega \subset \mathbb{R}^n$: convex domain, $\quad -\infty \le \alpha \le \infty$

u : nonnegative func. in Ω

Then u is α-concave in Ω if

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$\Omega \subset \mathbb{R}^n$: convex domain, $-\infty \le \alpha \le \infty$

u : nonnegative func. in Ω $\quad \left(P(u) = \{ x \in \Omega \mid u(x) > 0 \} \right)$

Then u is α-concave in Ω if

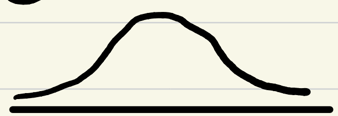
$$\begin{cases} P(u) \text{ is convex and } u \equiv \text{const. in } P(u) & (\alpha = \infty) \\ u^\alpha \text{ is concave in } P(u) & (\alpha > 0) \\ \log u \text{ is concave in } P(u) & (\alpha = 0) \\ u^\alpha \text{ is convex in } P(u) & (\alpha < 0) \\ \{ x \in \Omega : u(x) > \lambda \} \text{ is convex for } \lambda > 0 & (\alpha = -\infty) \end{cases}$$

Then u is α-concave if

$$\{u>0\} = P(u)$$

$$\begin{cases} P(u) \text{ is convex and } u \equiv \text{const. in } P(u) & (\alpha = \infty) \\ u^{\alpha} \text{ is concave in } P(u) & (\alpha > 0) \\ \log u \text{ is concave in } P(u) & (\alpha = 0) \quad \text{log-concavity} \\ u^{\alpha} \text{ is convex in } P(u) & (\alpha < 0) \\ \{x \in \Omega : u(x) > \lambda\} \text{ is convex for } \lambda > 0 & (\alpha = -\infty) \quad \text{quasi-concavity} \end{cases}$$

* $e^{-|x|^2}$: 0-concave, $\quad \dfrac{1}{1+|x|^2}$: $-\dfrac{1}{2}$-concave



Then u is α-concave if $\qquad \{u>0\}$

$\shortparallel$

↑ strong

$$\begin{cases} P(u) \text{ is convex and } u \equiv \text{const. in } P(u) & (\alpha = \infty) \\ u^{\alpha} \text{ is concave in } P(u) & (\alpha > 0) \\ \log u \text{ is concave in } P(u) & (\alpha = 0) \quad \text{log-concavity} \\ u^{\alpha} \text{ is convex in } P(u) & (\alpha < 0) \\ \{x \in \Omega : u(x) > \lambda\} \text{ is convex for } \lambda > 0 & (\alpha = -\infty) \quad \text{quasi-concavity} \end{cases}$$

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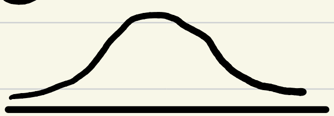

* u: α-concave $\Longrightarrow$ u: β-concave
$\beta \le \alpha$

§2. Concavity and PDEs

(1) Brascamp–Lieb '76

ϕ is log-concave in $\mathbb{R}^n$ $\Rightarrow$ $e^{t\Delta}\phi$ is log-concave in $\mathbb{R}^n$ for $t>0$.

(2) Korevaar '83

Concavity Maximum principle

(a) Ω: convex domain in $\mathbb{R}^n$

$$u: \text{sol. of } \begin{cases} \partial_t u = \Delta u & \text{in } \Omega\times(0,\infty), \\ u = 0 & \text{on } \partial\Omega\times(0,\infty) \text{ if } \partial\Omega\neq\emptyset, \\ u(x,0)=\phi(x) & \text{in } \Omega. \end{cases}$$

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Preservation of log-concavity by the Dirichlet heat flow

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Preservation of log-concavity by the Dirichlet heat flow

(b) Ω : bounded convex domain in $\mathbb{R}^n$

1st Dirichlet eigenfunction for $-\Delta$ in Ω is log-concave in Ω.

(3) Kennington '85, Kawohl '85

Greco–Kawohl '99
Alvarez–Lasry–Lions '97
Lee–Vazquez '03, '08
I–Salani '08 ~

Ω : convex domain in $\mathbb{R}^n$

(a) $-\Delta u = f(x) \geq 0$ in Ω, $u=0$ on $\partial\Omega$

f : q-concave in Ω with $q \geq 1$ $\Rightarrow$ u : $\dfrac{q}{1+2q}$-concave in Ω

(b) $-\Delta u = 1$ in Ω, $u=0$ on $\partial\Omega$ (torsion problem)

u : $\frac{1}{2}$-concave in Ω ($\sqrt{u}$ is concave in Ω)

(c) $-\Delta u = u^\gamma$ in Ω, $u=0$ on $\partial\Omega$
$(0<\gamma<1)$

u : $\dfrac{1-\gamma}{2}$-concave in Ω.

§ 3 Dirichlet heat flow and quasi-concavity

Brascamp–Lieb '76, Korevaar '83

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Preservation of log-concavity by the Dirichlet heat flow

§ 3 Dirichlet heat flow and quasi-concavity

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Preservation of log-concavity by the Dirichlet heat flow

Q1 Is quasi-concavity preserved by the Dirichlet heat flow?

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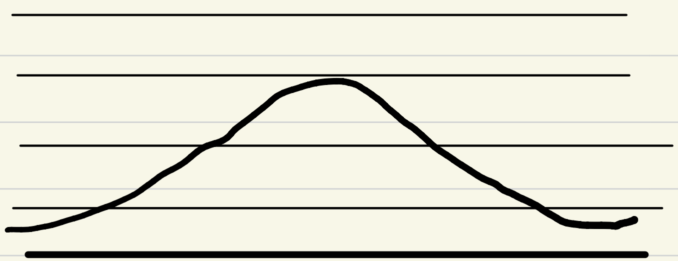
Answer $n=1 \Rightarrow$ Yes, $n \geq 2 \Rightarrow$ No

(I–Salani '08)

⑩ $n=1$



Roughly speaking,

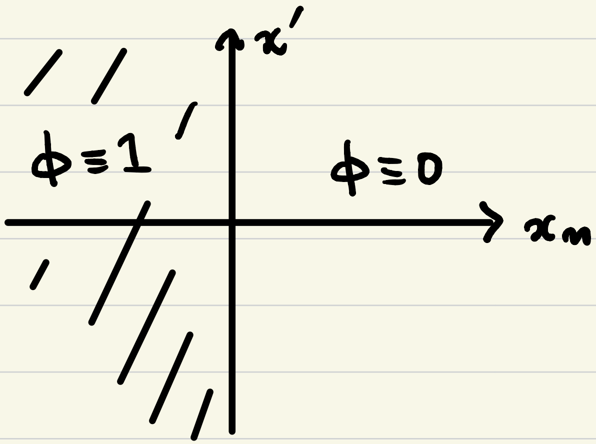
$u(\cdot,t)$ is quasi-concave

$\Updownarrow$

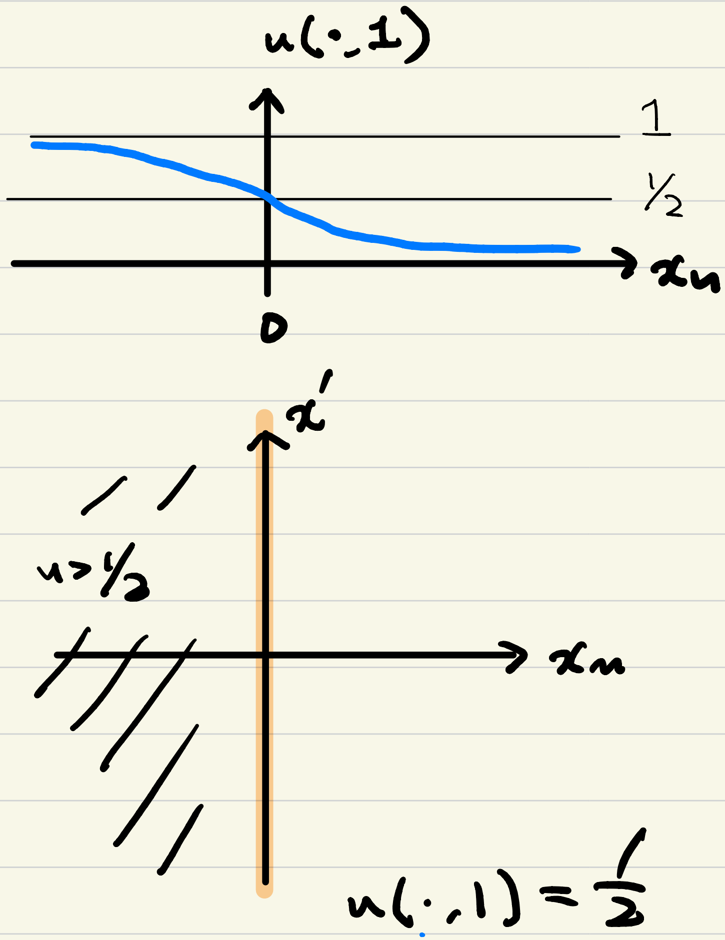
this is kept by 1-dim Dirichlet heat flow. $\Rightarrow$ (Intersection numbers of u and constant functions are 0, 1 or 2.

(iii) $n \geq 2$

(1) $\phi(x) = \chi_{\{x_n < 0\}}$



$\Longrightarrow$



(iii) $n \geq 2$ $\quad u = e^{t\Delta}\phi$

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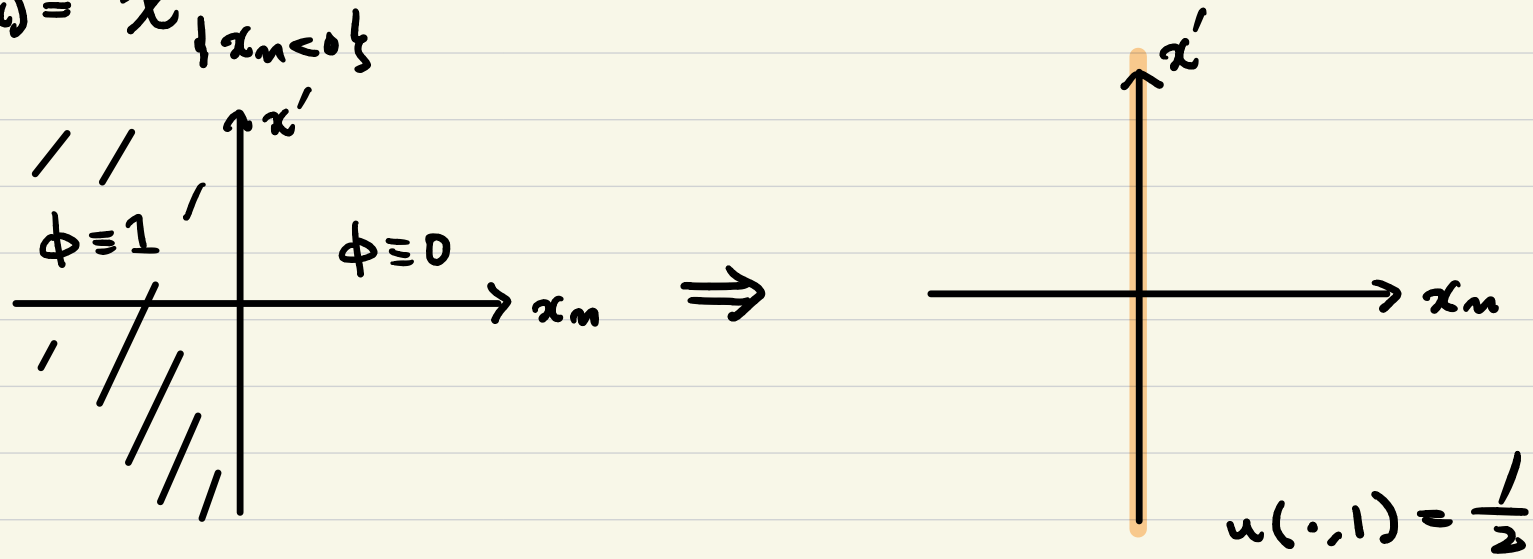
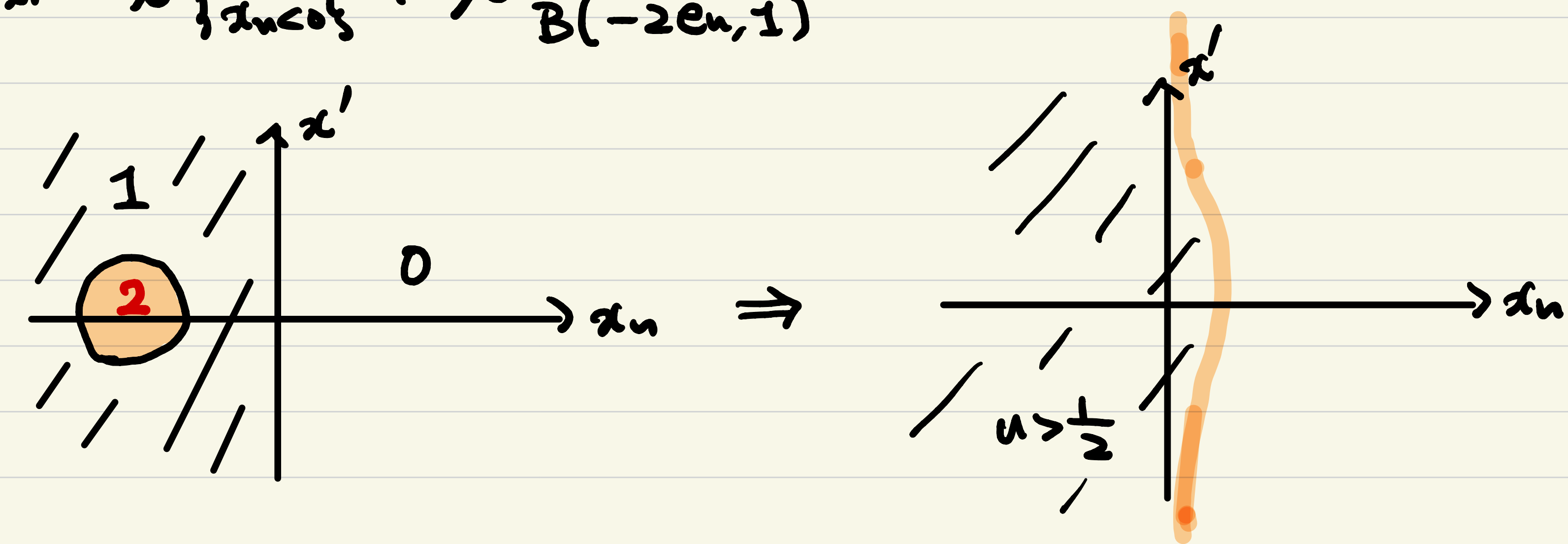


(2) $\phi(x) = \chi_{\{x_n<0\}} + \chi_{B(-2e_n,1)}$



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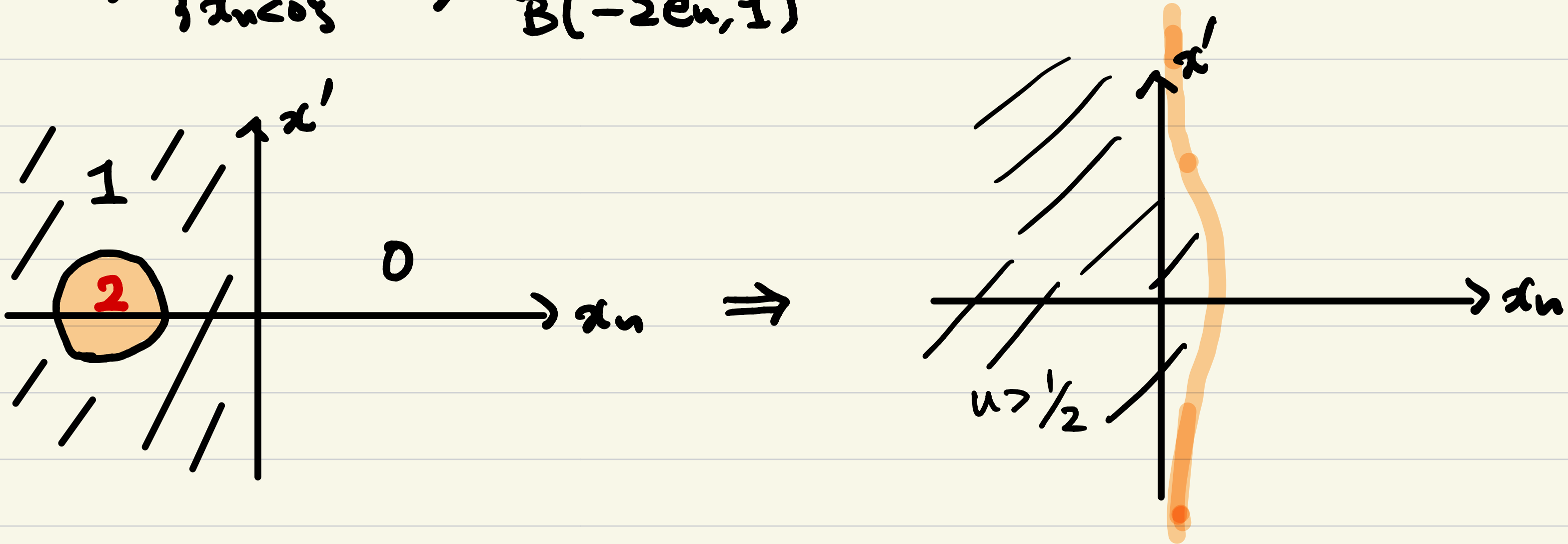

* $\{x_n<0\} \subset \{u(1) > \frac{1}{2}\} \neq$ half space

* If a convex set includes a half space, then the convex set must be a half space or $\mathbb{R}^n$.

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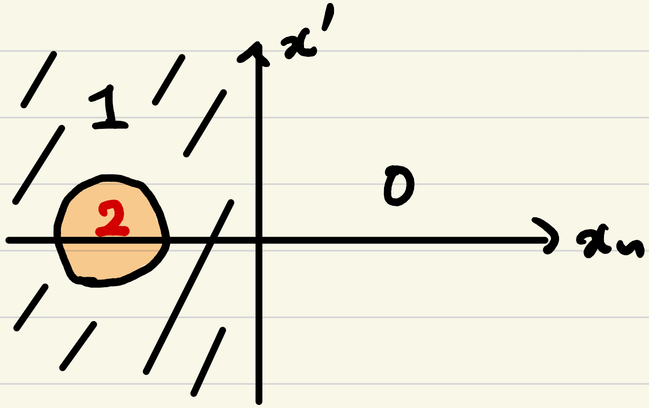
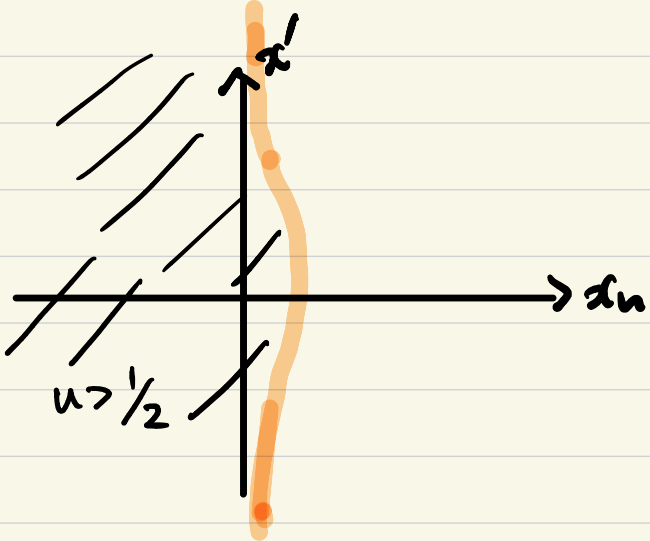


$\Rightarrow$



* $\{x_n<0\} \subset \{u(1) > \frac{1}{2}\} \neq$ half space

* If a convex set includes a half space, then the convex set must be a half space or $\mathbb{R}^n$.

$\Rightarrow \{u(1) > \frac{1}{2}\}$ is not convex $\Rightarrow$ $\hat{e}\phi$ is not quasi-concave in $\mathbb{R}^n$

(Asymmetry breaks the convexity of a level set.)

Q1 Is quasi-concavity preserved by the Dirichlet heat flow?

Answer $n=1 \Rightarrow$ Yes, $n\geq 2 \Rightarrow$ No

Q2 What is the weakest (strongest) concavity preserved by the Dirichlet heat flow?

log-concavity?

Q3 When is the quasi-concavity preserved by the Dirichlet heat flow?

§ 4. Main result (with P. Salani & A. Takatsu)

Theorem A

$\Omega \subset \mathbb{R}^n$: convex domain, $-\infty < \alpha < 0$, $n \geq 2$

Then $\exists \phi$: α-concave in Ω,

$\exists T > 0$

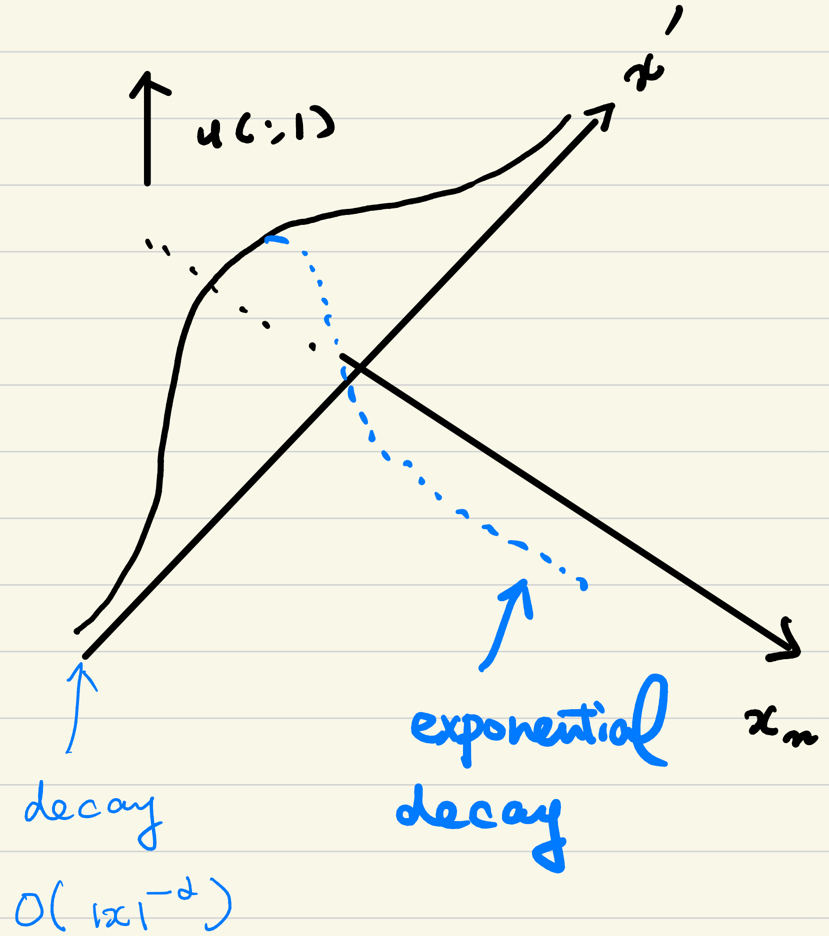
s.t. the sol. u of
$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \text{ if } \partial\Omega \neq \phi \\ u(0) = \phi & \text{in } \Omega \end{cases}$$

is not quasi-concave at $t = T$. //

Corollary Negative power concavity is not preserved by the Dirichlet heat flow.

Proof Let $-\infty < \alpha < 0$.

Set $\phi(x', x_n) = (1+|x|^2)^{-\frac{1}{2\alpha}} \chi_{\{x_n<0\}}$. ($\alpha$-concave in $\mathbb{R}^n$)



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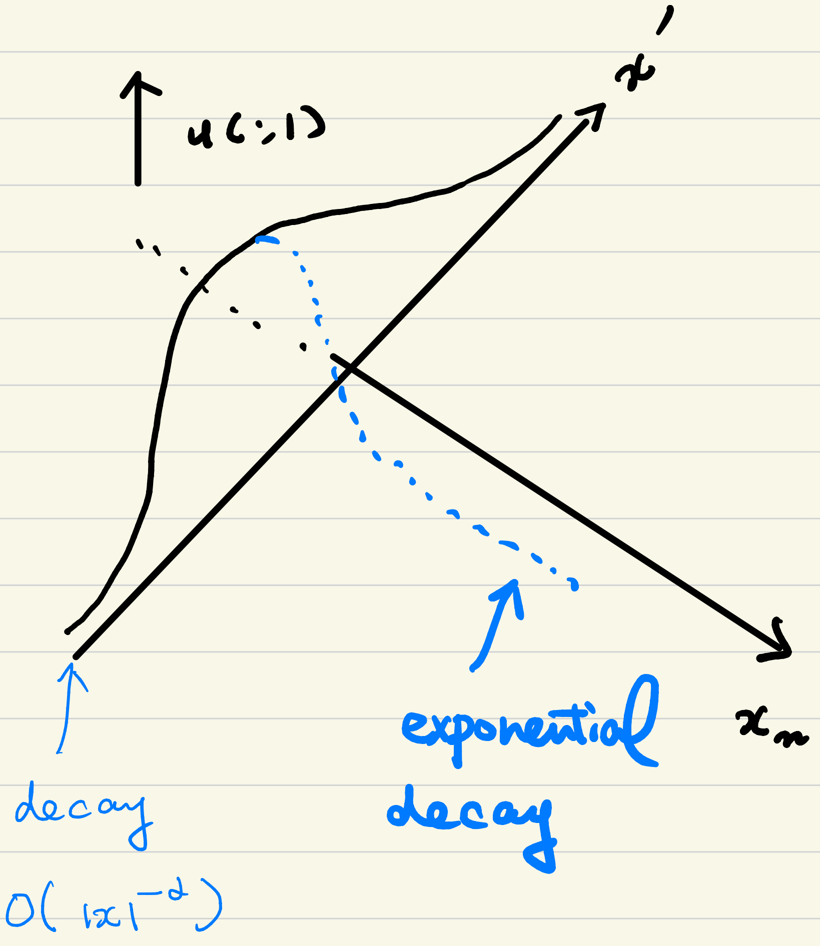
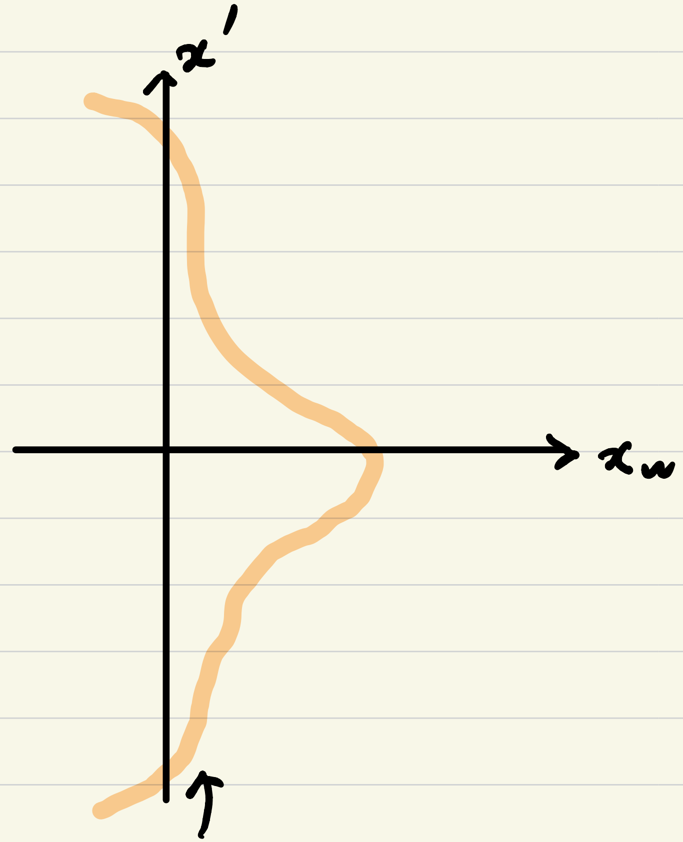
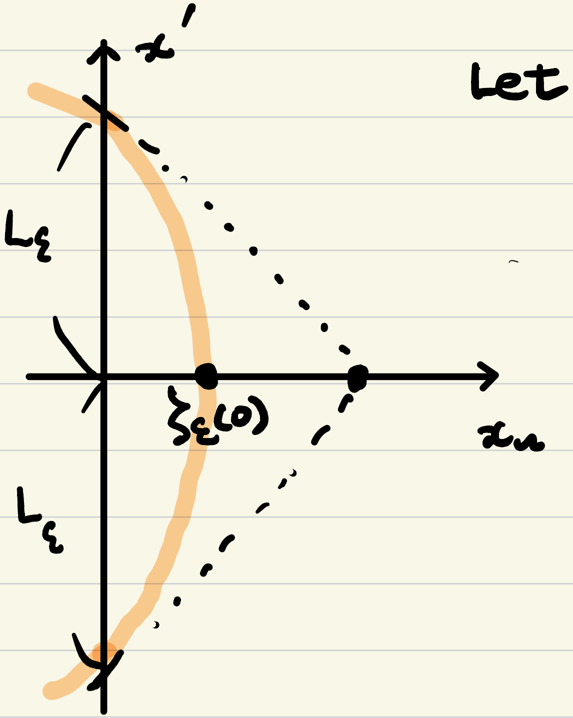


$\Rightarrow$



$\partial\{u(\cdot,1)>\varepsilon\}$ for $0<\varepsilon\ll 1$

Let $0 < \varepsilon << 1$ and assume that $\{ u(x', x_n, 1) > \varepsilon \}$ is convex.



$\partial \{ u(\cdot, 1) > \varepsilon \}$

Let

$$\zeta_\varepsilon : [0, L_\varepsilon] \longrightarrow [0, \infty)$$

s.t.

$$u(x', \zeta_\varepsilon(|x'|)) = \varepsilon.$$

(1) implicit func. thm.

$$|\zeta_\varepsilon'(|x'|)| = \frac{|\nabla_{x'} u(x', 0, 1)|}{|\partial_{x_n} u(x', 0, 1)|} \leq C L_\varepsilon^{-1}$$

$|x'| = L_\varepsilon$

(2)

$$\zeta_\varepsilon(0) \leq L_\varepsilon |\zeta_\varepsilon'(|x'|)| \leq C$$

contradiction

(3) explicit formula

$$\zeta_\varepsilon(0) \geq \sqrt{2 |\log \varepsilon / C|} \longrightarrow \infty \quad (\varepsilon \to 0)$$

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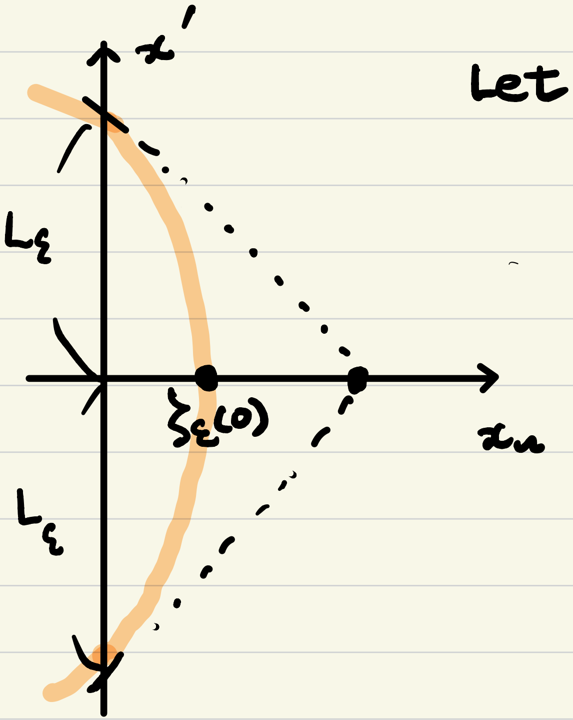

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//

$\partial \{ u(\cdot, 1) > \varepsilon \}$

is not convex.

$\Downarrow$

$u(\cdot, 1)$ is not

quasi-concave in $\mathbb{R}^n$.

Summary.

Q2 What is the weakest (strongest) concavity preserved by the Dirichlet heat flow?

Ans. log-concavity among power concavities.

Q3 When is the quasi-concavity preserved by the Dirichlet heat flow?

Ans. If ϕ is negative power concave, then the quasi-concavity is not necessarily preserved by the Dirichlet heat flow.