Yamabe flow on some singular spaces.

Joint work with :
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[Submitted on 3 Jun 2021]

Convergence of the Yamabe flow on singular spaces with positive Yamabe constant

Gilles Carron, Jørgen Olsen Lye, Boris Vertman

In this work, we study the convergence of the normalized Yamabe flow with positive Yamabe constant on a class of pseudo-manifolds that includes stratified spaces with iterated cone-edge metrics. We establish convergence under a low energy condition. We also prove a concentration - compactness dichotomy, and investigate what the alternatives to convergence are. We end by investigating a non-convergent example due to Viaclovsky in more detail.

Plane of the talk

- Yamabe flow on the smooth setting

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- Asymptotic behavior of Yamabe flow

Yamabe flow

On closed manifold, the Yamabe flow [R. Hamilton,1989] is the gradient flow of the Hilbert functional

$$g \mapsto \int_M \mathrm{Scal}_g \mathrm{dv}_g$$

$$\text{on } \mathcal{C}(g_0) := \left\{ g = e^{2f} g_0, \mathrm{vol}_g(M) = \int_M \mathrm{dv}_g = 1 \right\}.$$

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The question was to solve by parabolic method the Yamabe problem

$$\text{Given } g_0, \text{ find } g \in e^{2f} g_0 \text{ with } \mathrm{Scal}_g = \mathrm{C} \text{ and } \mathrm{vol}_g(M) = 1.$$

Existence of Yamabe metric on closed smooth manifold

Introducing $Y(M, g_0) = \inf_{g \in \mathcal{C}(g_0)} \int_M \mathrm{Scal}_g \mathrm{dv}_g$, we know that there is $g = e^{2f} g_0$ such that

$$\mathrm{Scal}_g = Y(M, g_0) \text{ and } \mathrm{vol}_g(M) = 1.$$

This problem has been solved by Yamabe, Trudinger, Aubin and Schoen. As an abstract of the story, we have

- (Aubin, 1974) : we always have $Y(M, g_0) \leq Y(\mathbb{S}^n, [\mathrm{rounded}])$.

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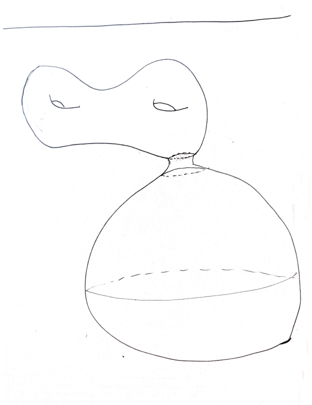
- (Aubin, 1974, Schoen 1984) : if $Y(M, g_0) = Y(\mathbb{S}^n, [\mathrm{rounded}])$ then $M = \mathbb{S}^n$ and $g_0 = e^{-2f}[\mathrm{rounded}]$.

Convergence of the Yamabe flow on the smooth setting : a brief summary

- The Yamabe flow exists of all times,
- There is either convergence or concentration (formation of bubbles).
- Positive mass theorem prevents the formation of bubble.

Mostly from [Schwetlick- Struwe, 2003] and [Brendle, 2005 & 2007].



Some stratified space : the surface of a cube

We are looking for the geometry of the surface of a cube :

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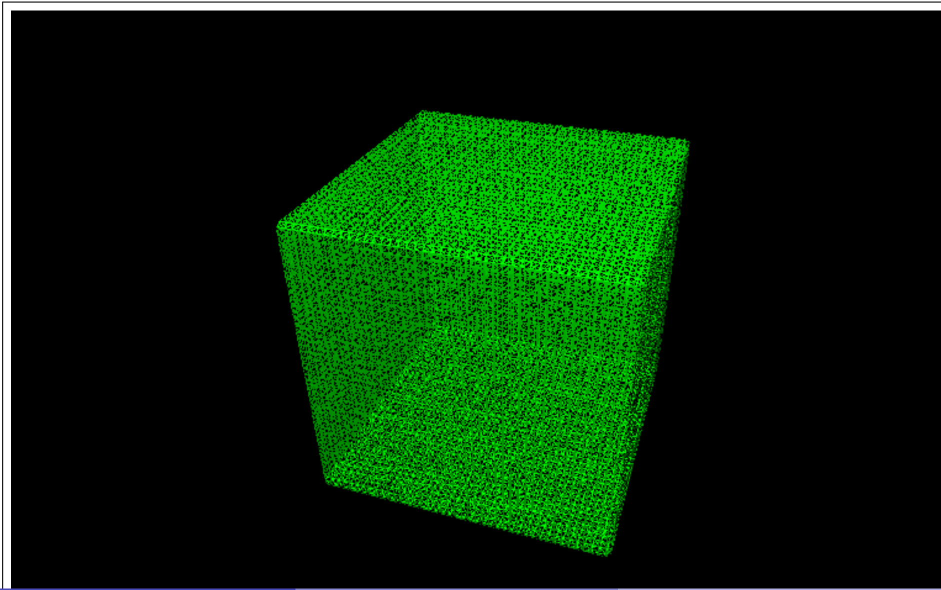

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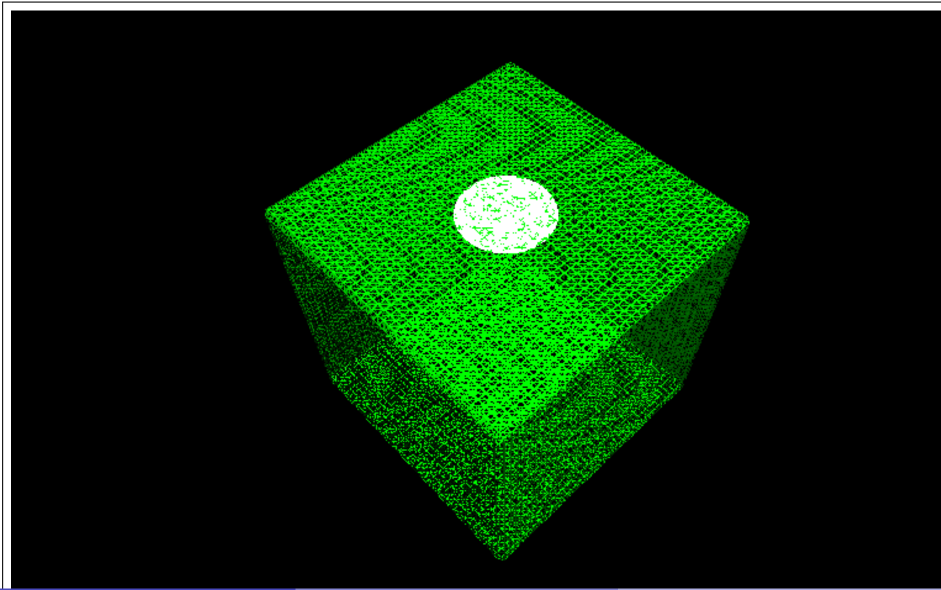

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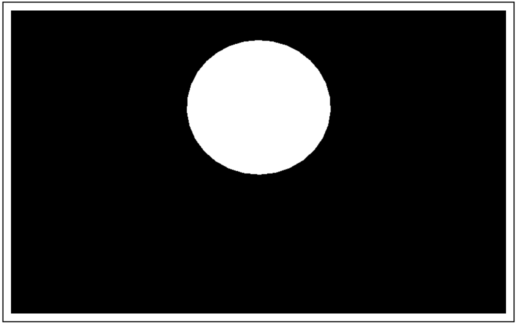

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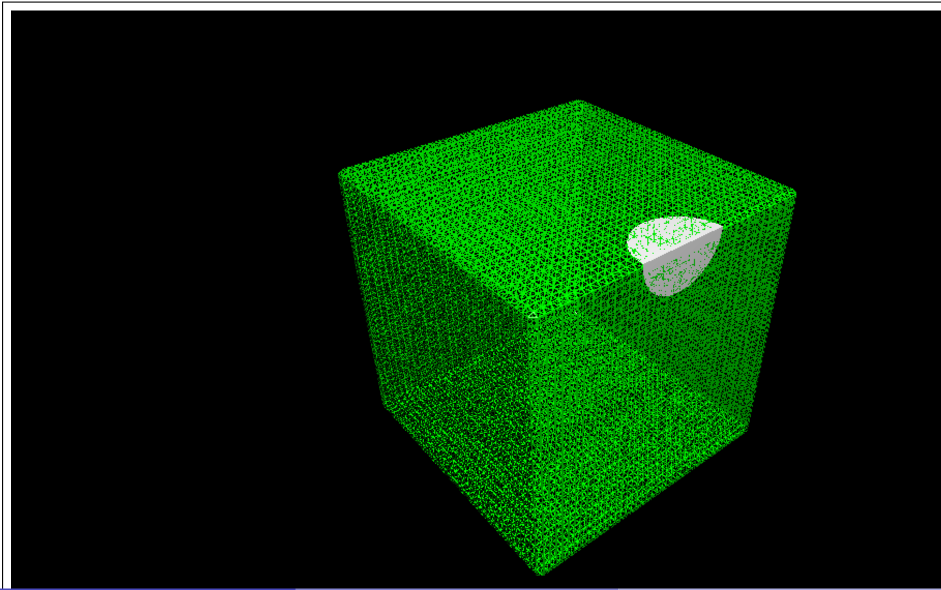

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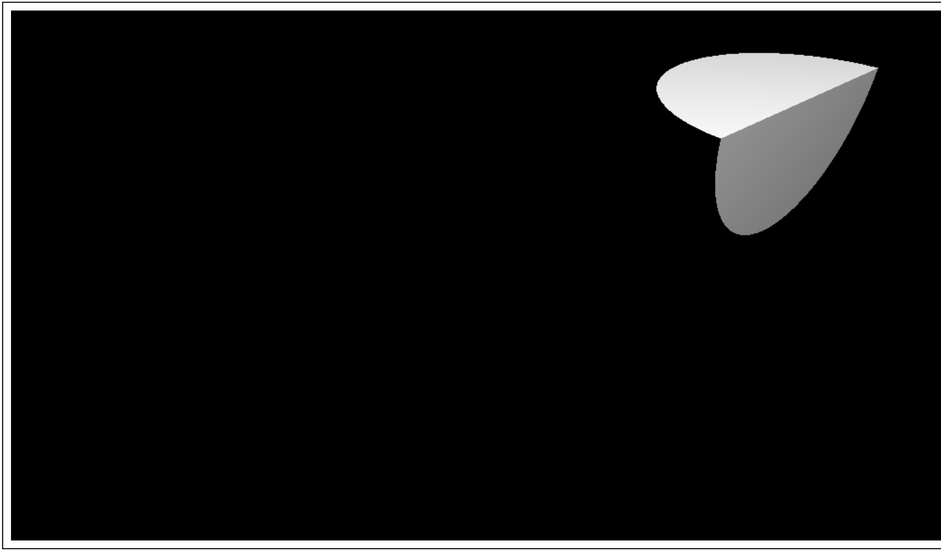

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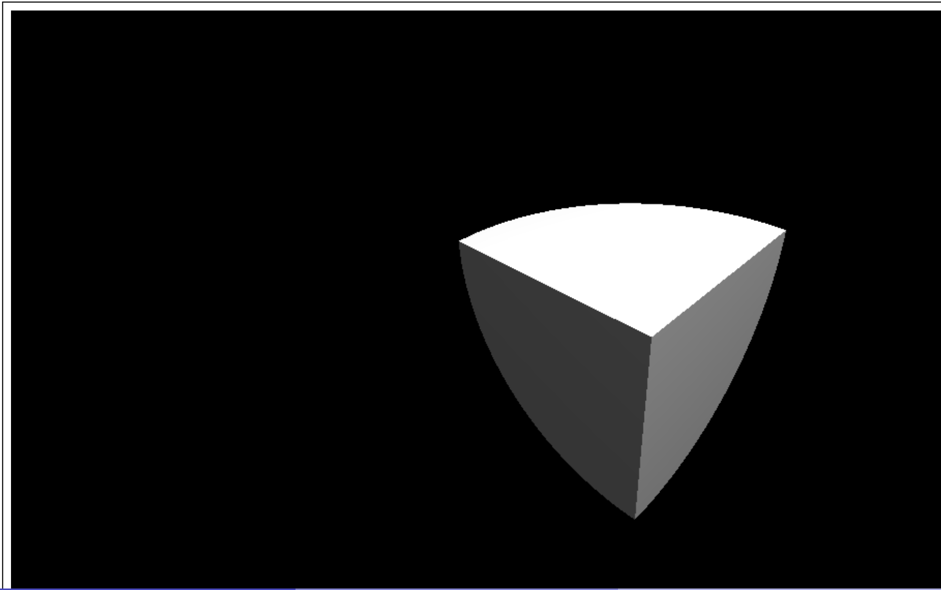

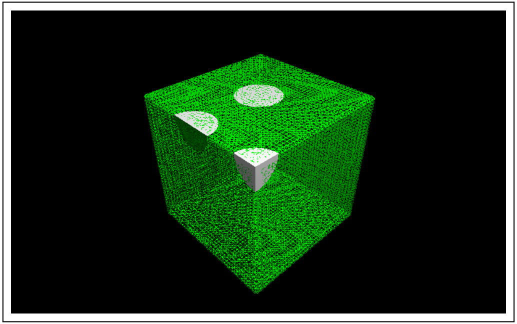

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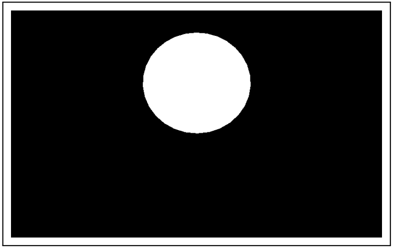
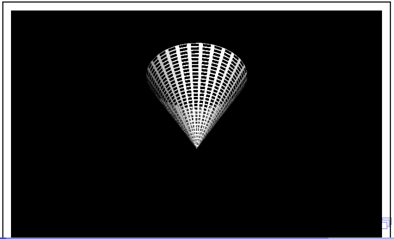
Summary : the surface of a cube has a decomposition $X \supset X_0$, where

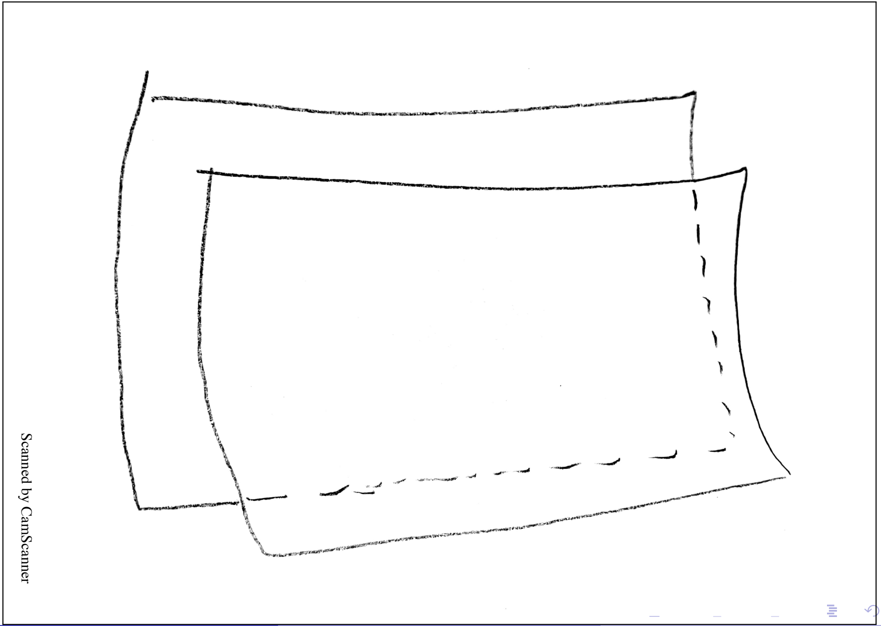
- near each point of $X \setminus X_0$, the geometry is Euclidean



- X_0 is the collection of 8 vertex and near each of these point the geometry is a cone over a circle of length $3\pi/2$



Some stratified space : the surface of a pillow



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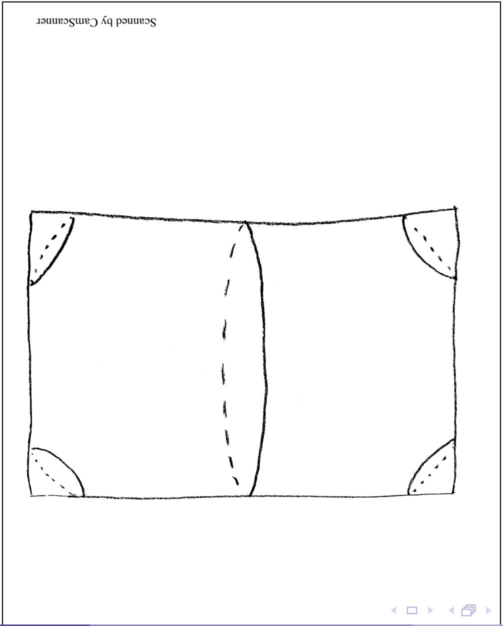

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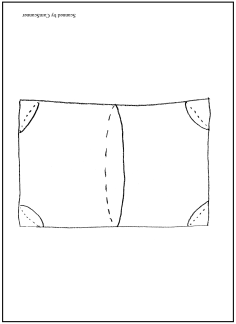

Summary : the surface of a pillow has a decomposition $X \supset X_0$, where

- near each point of $X \setminus X_0$, the geometry is Euclidean
- X_0 is the collection of 4 vertex and near each of these point the geometry is a cone over a circle of length π

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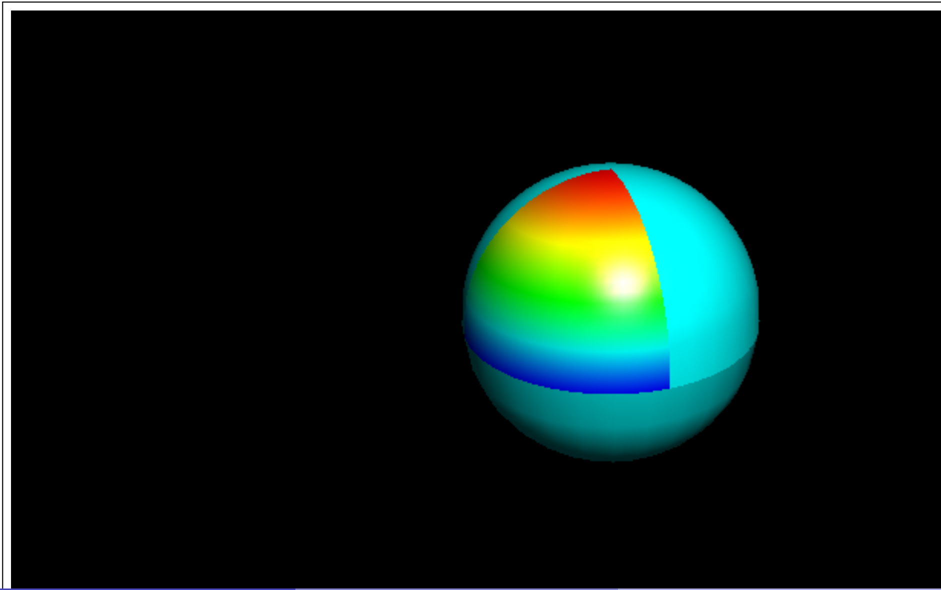

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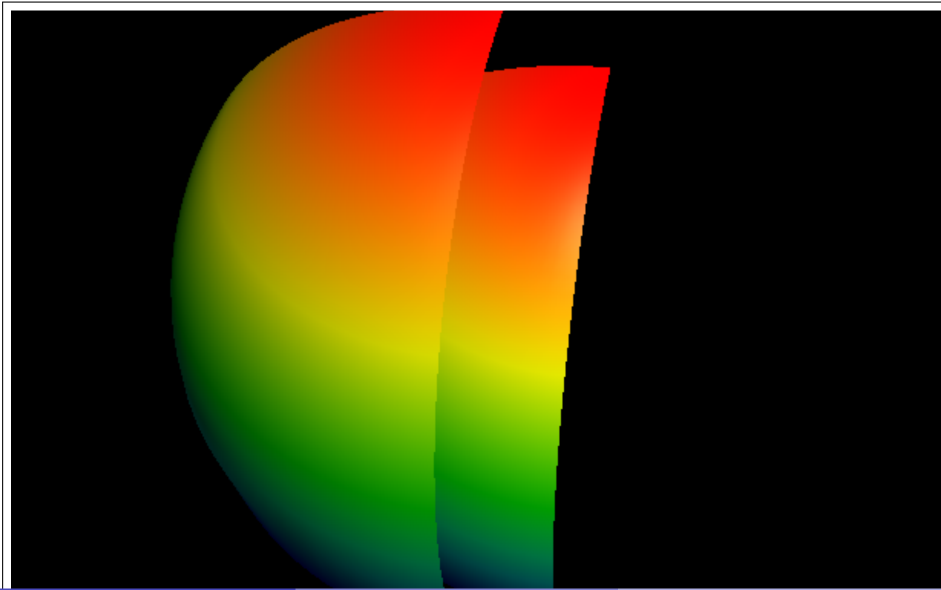

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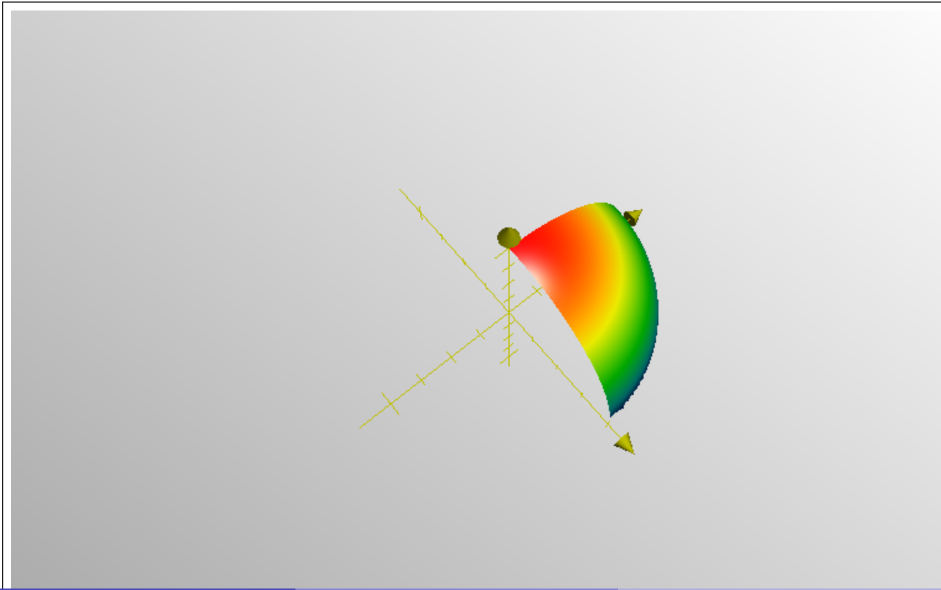

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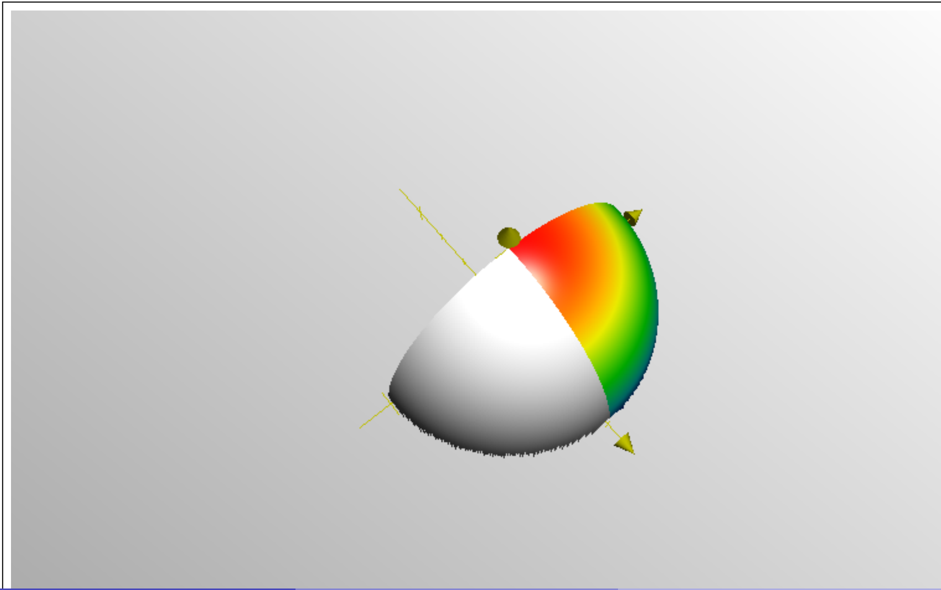

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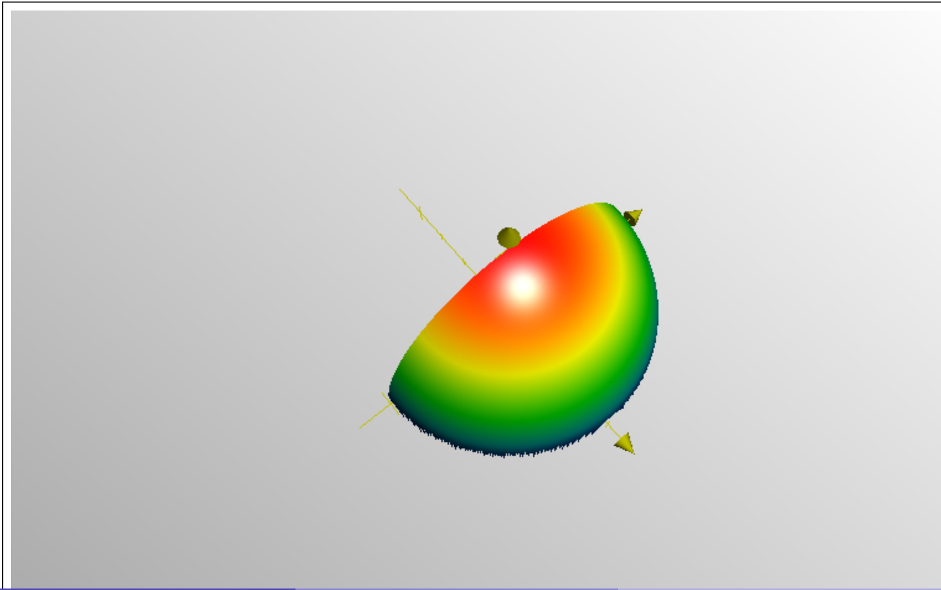

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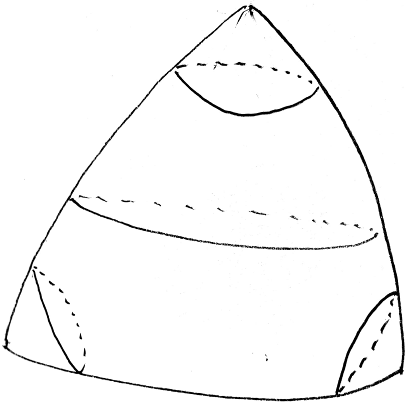

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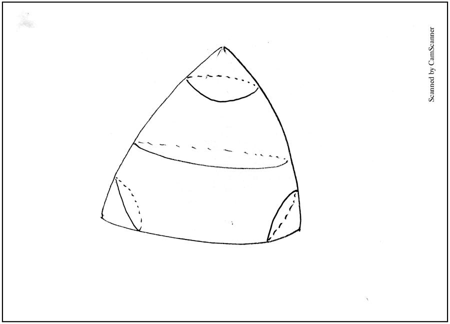
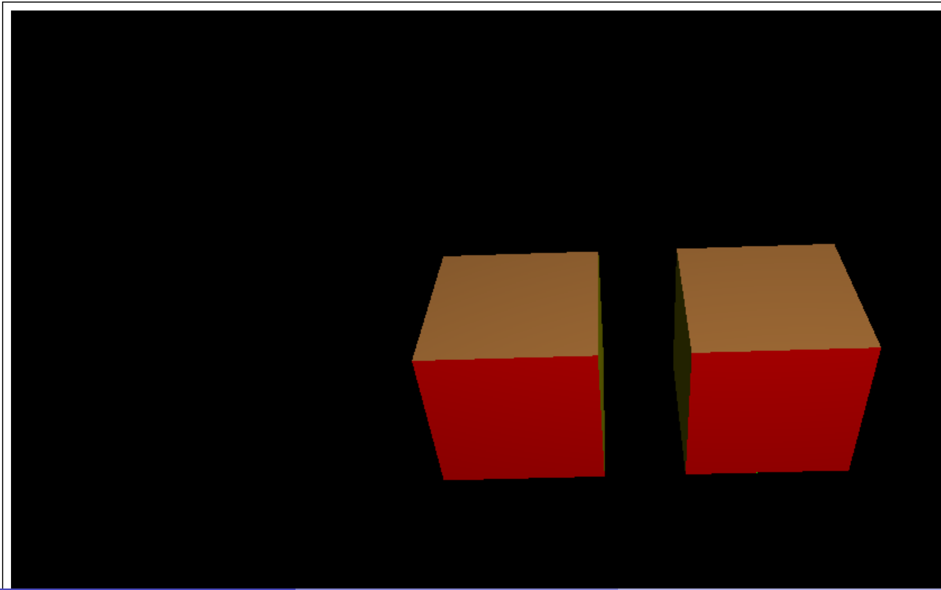


Summary : the surface of a berlingot has a decomposition $X \supset X_0$, where

- near each point of $X \setminus X_0$, the geometry is Spherical (Riemannian)
- X_0 is the collection of 3 vertex and near each of these points the geometry is a cone over a circle of length π

Some stratified space : the double solid cube



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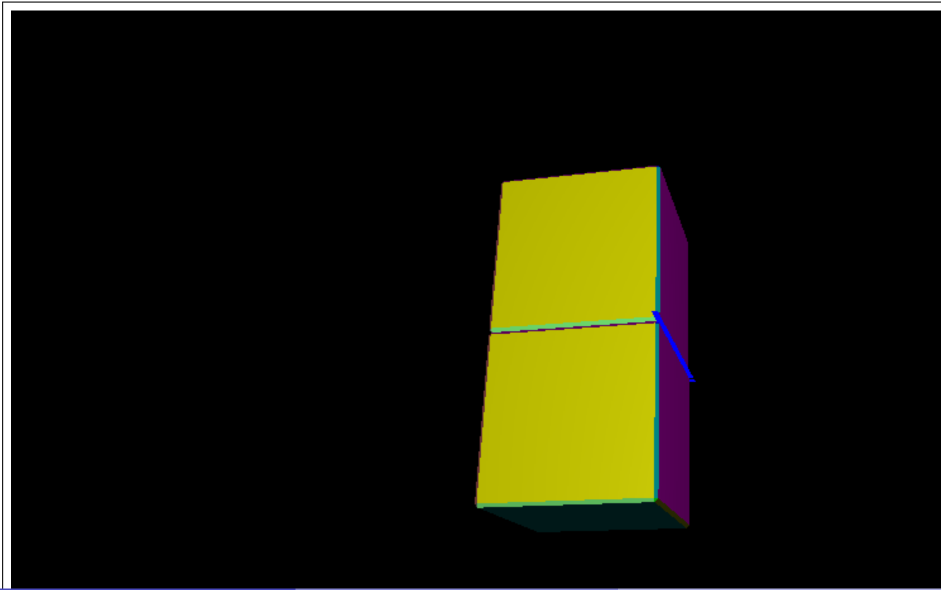

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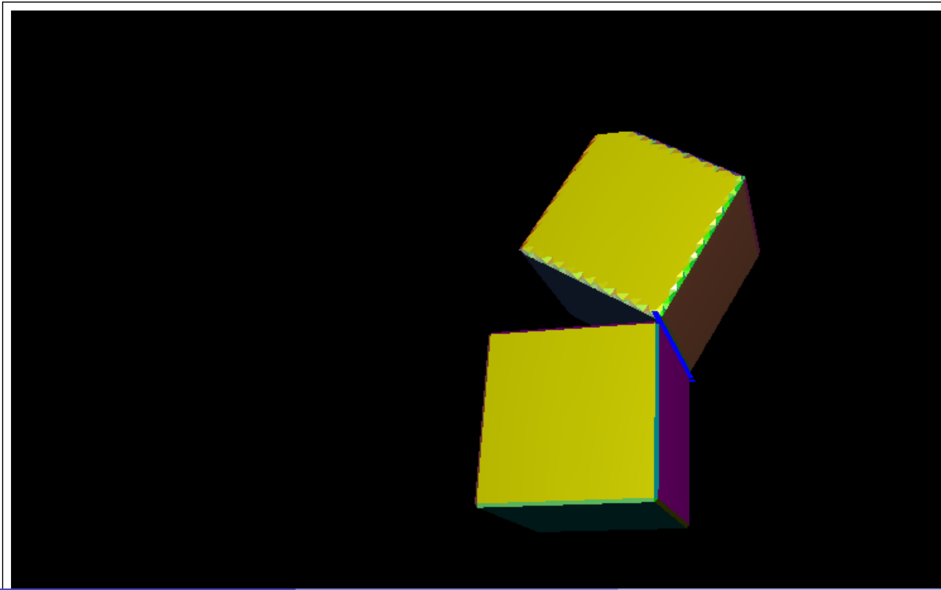

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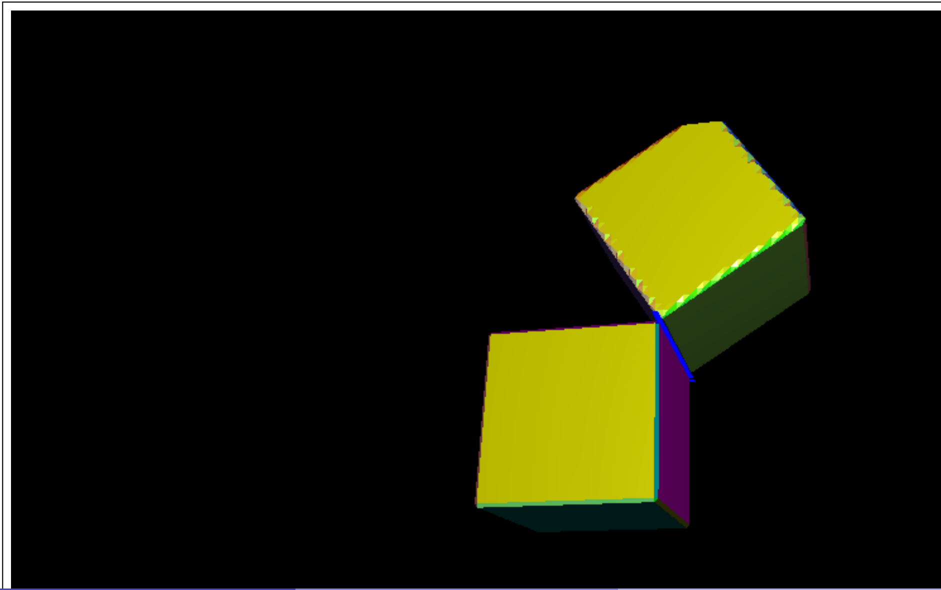

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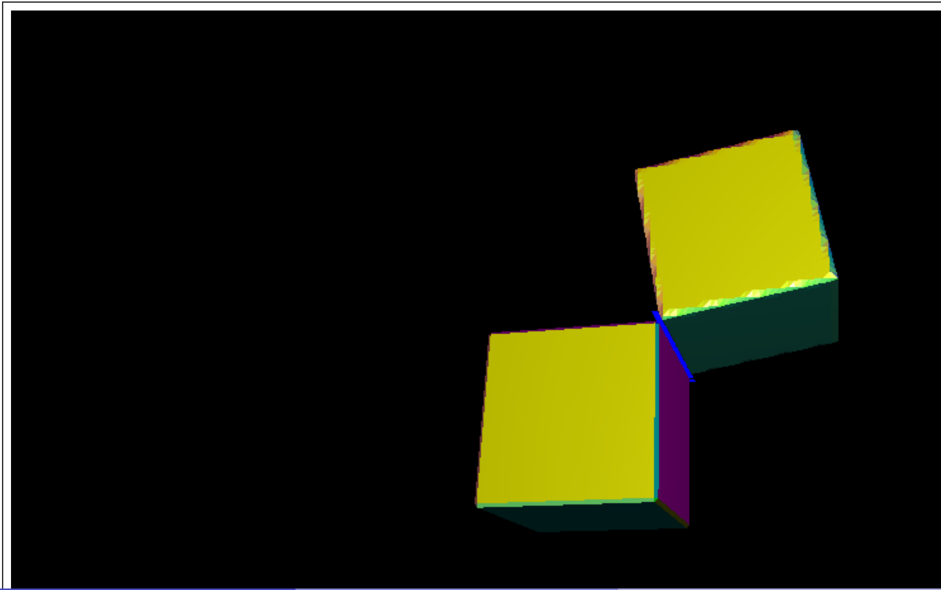

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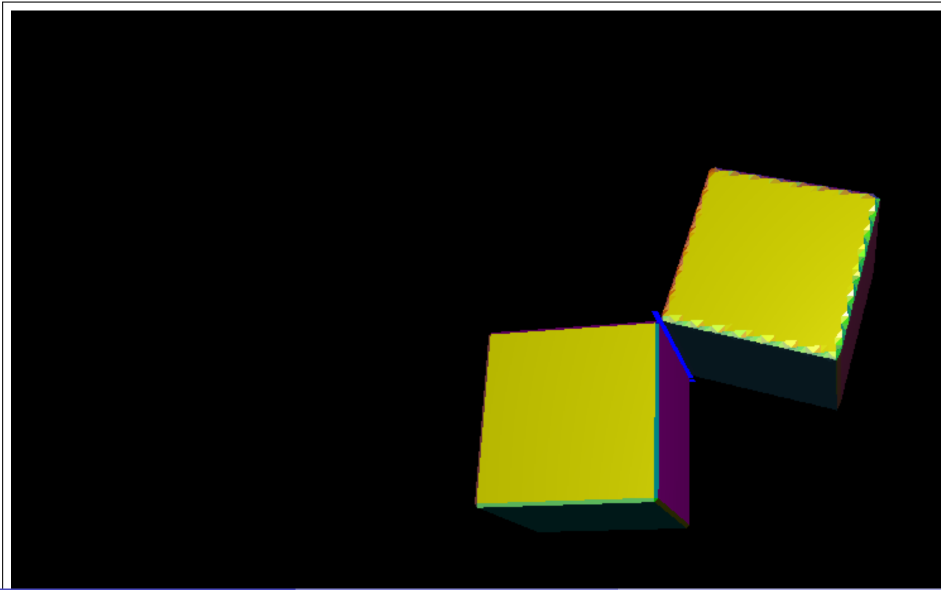

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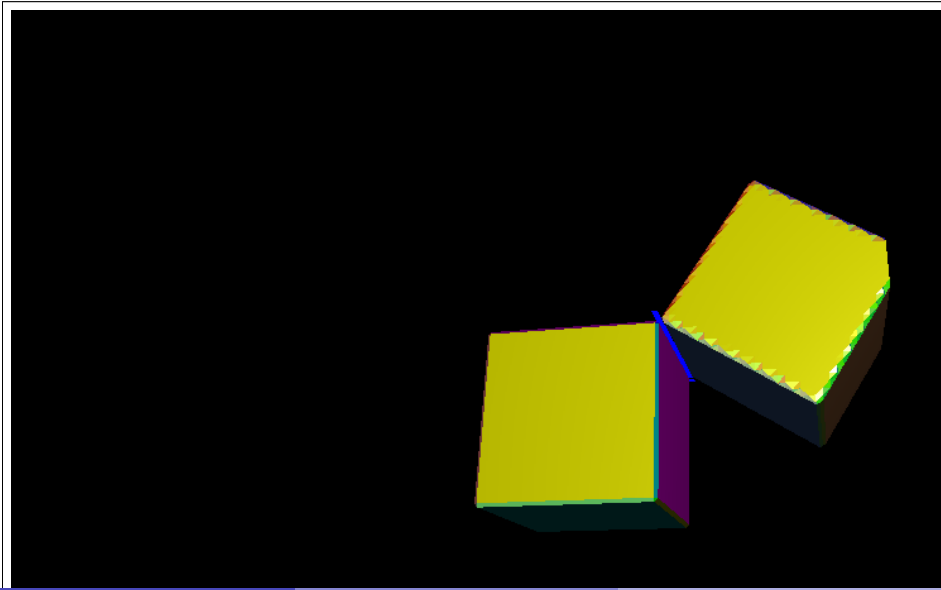

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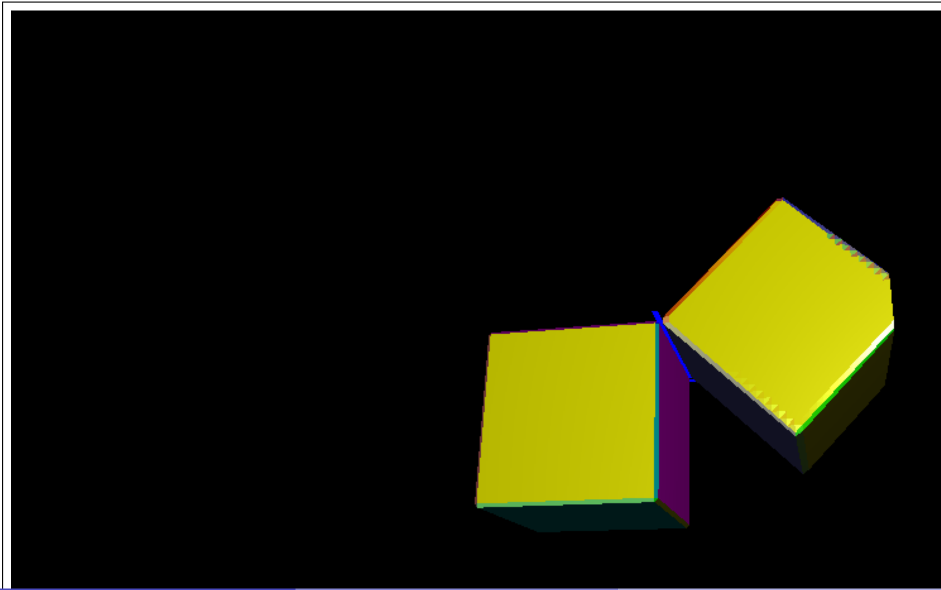

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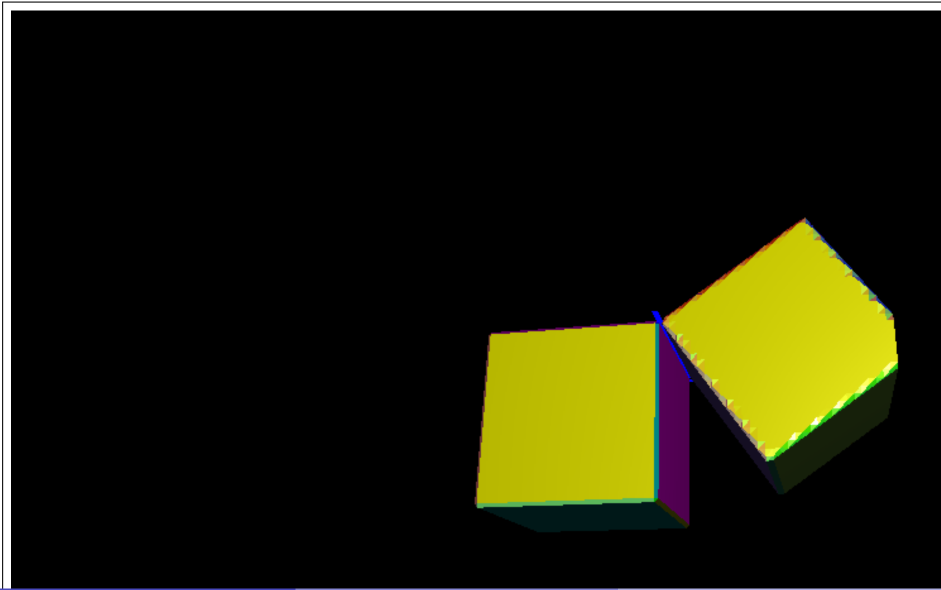

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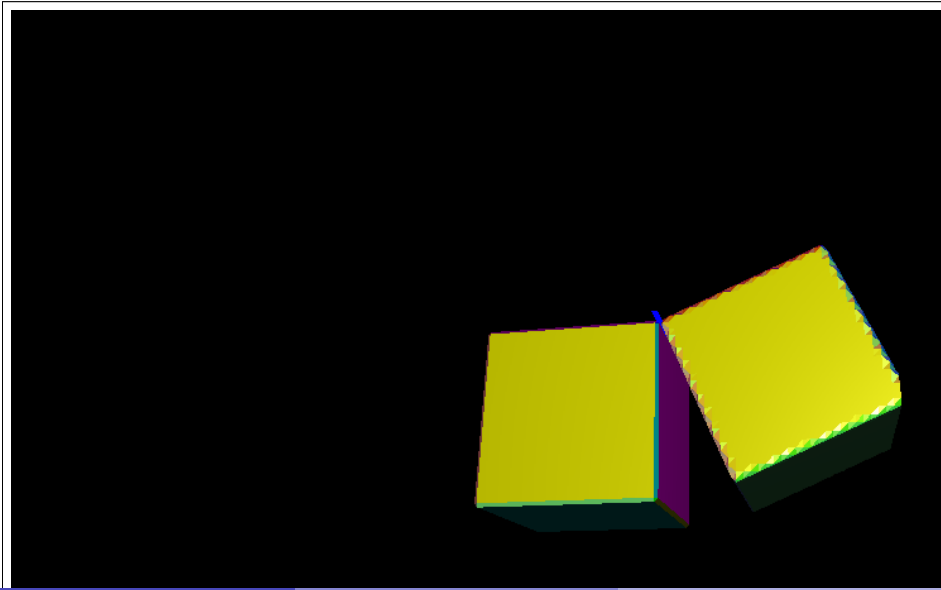

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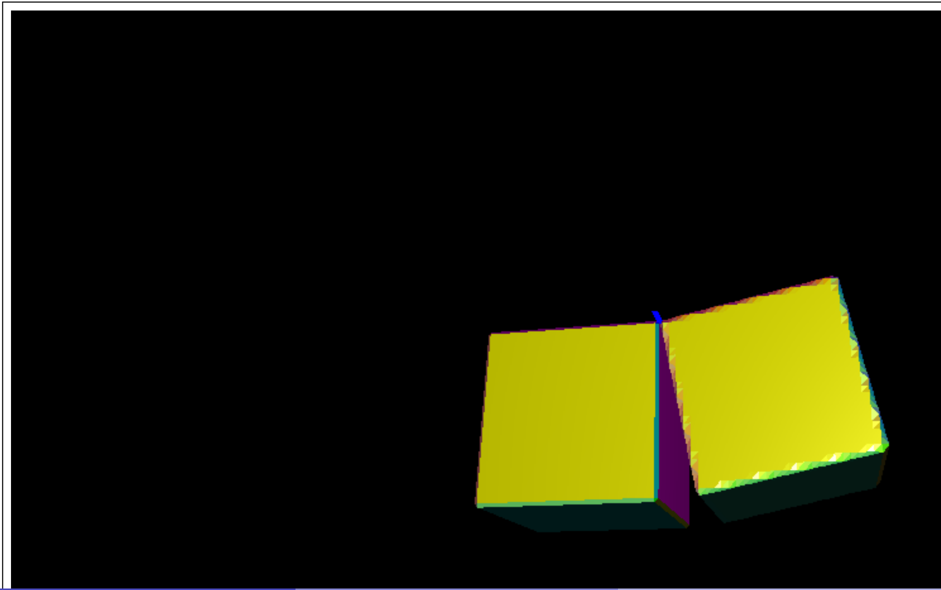

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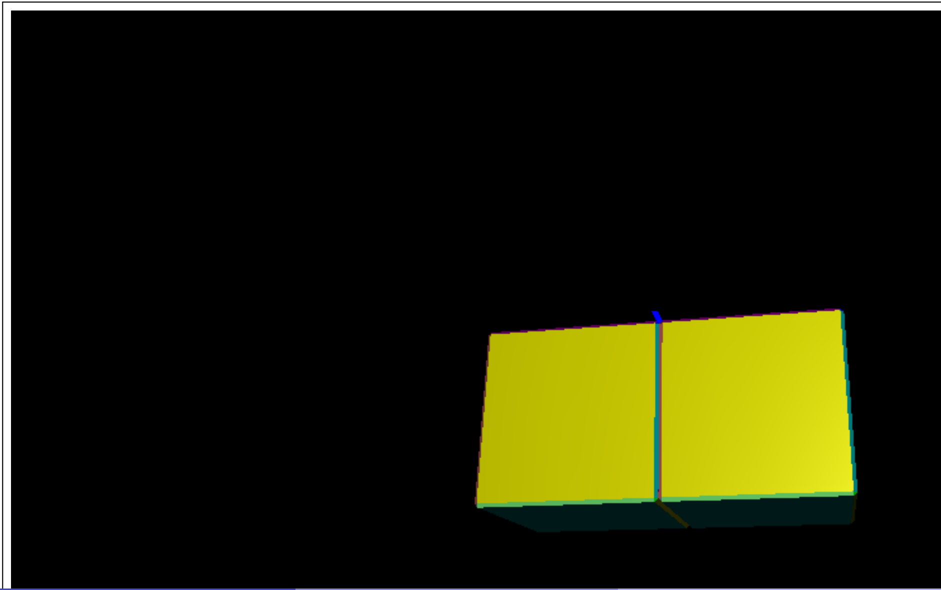

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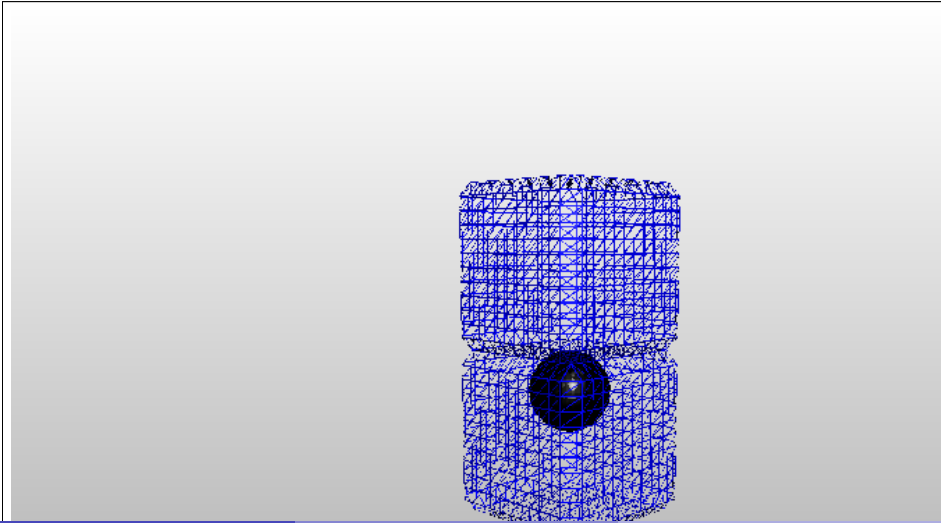
The geometry of the double solid cube is the following : at a point interior or on a face of the cube, the geometry is Euclidean.



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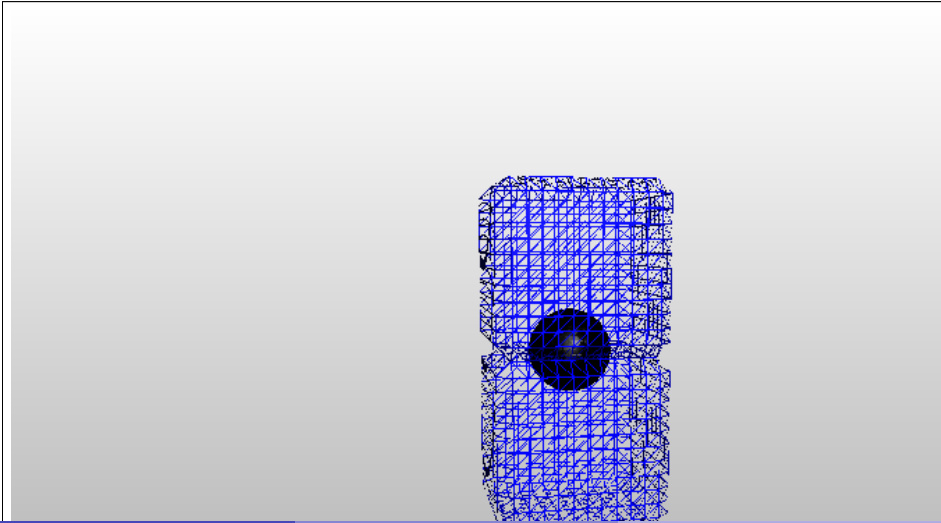

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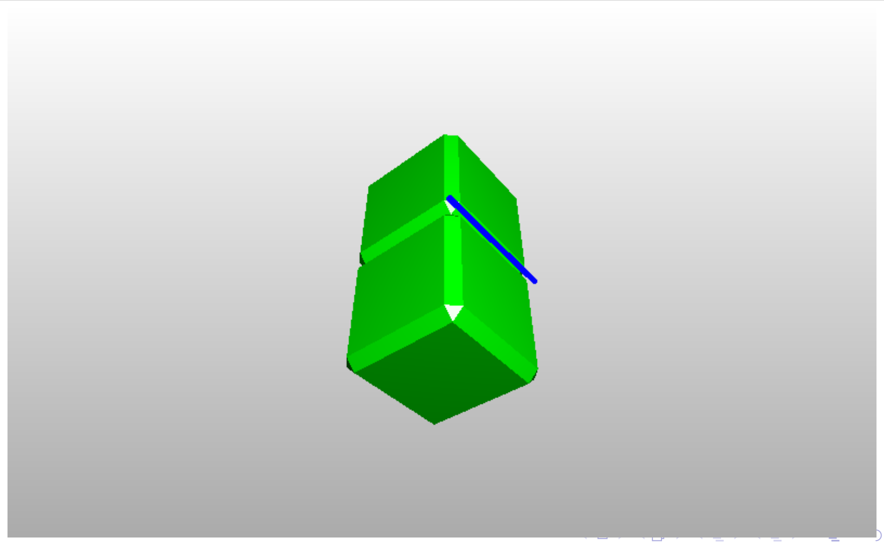

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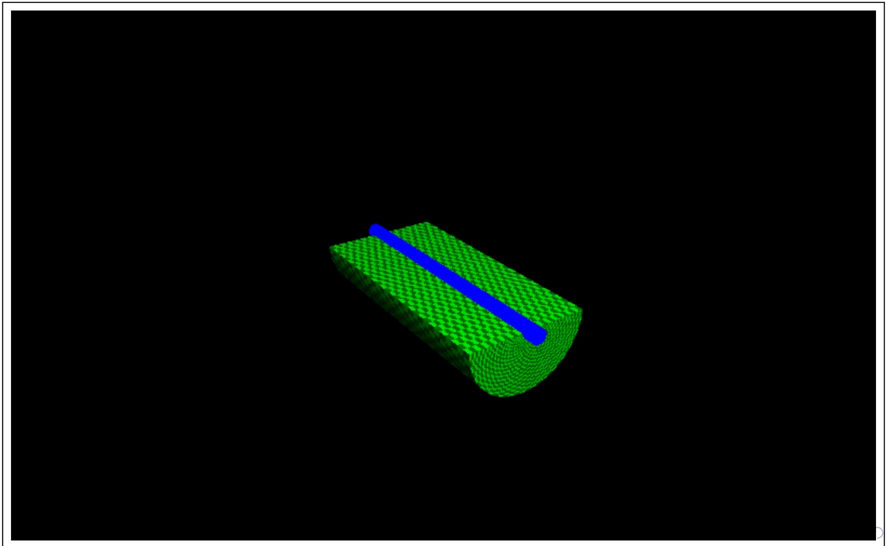

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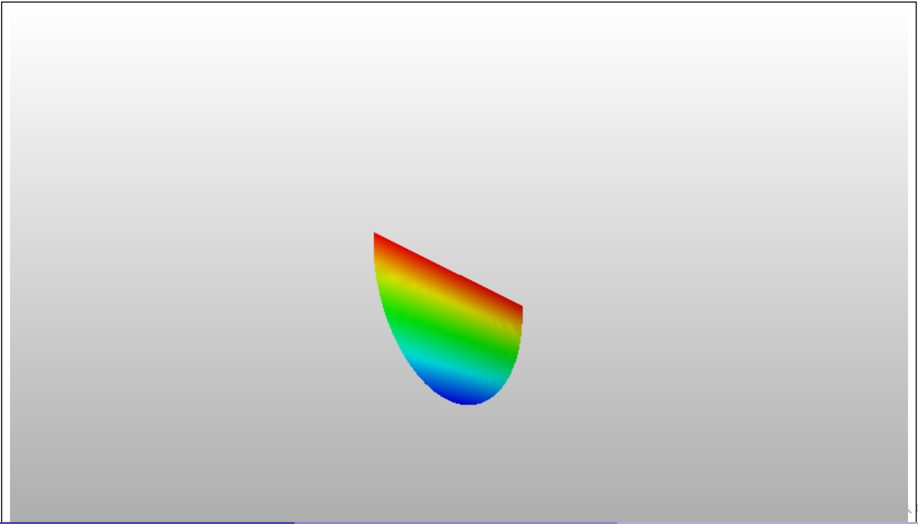

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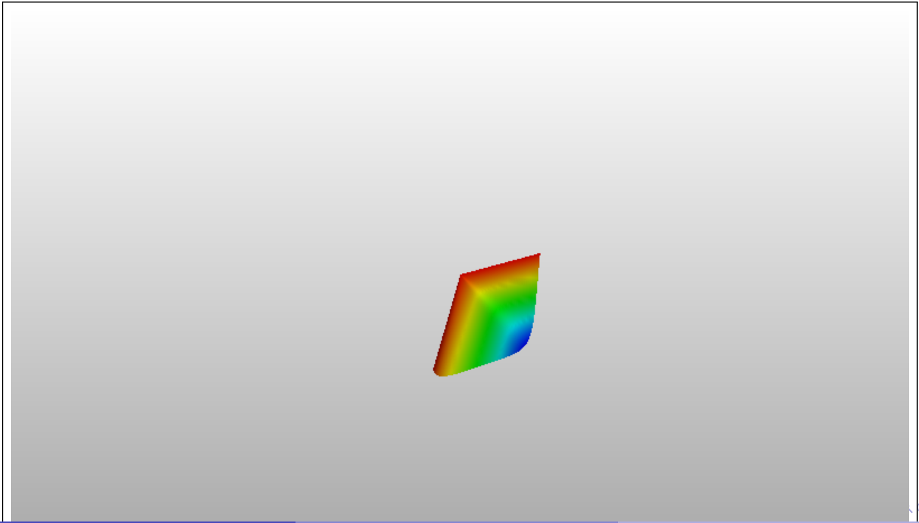

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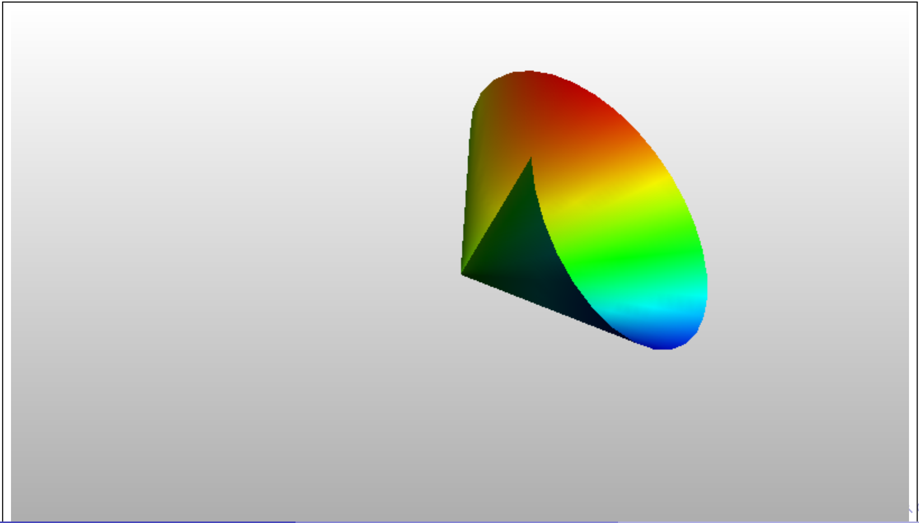

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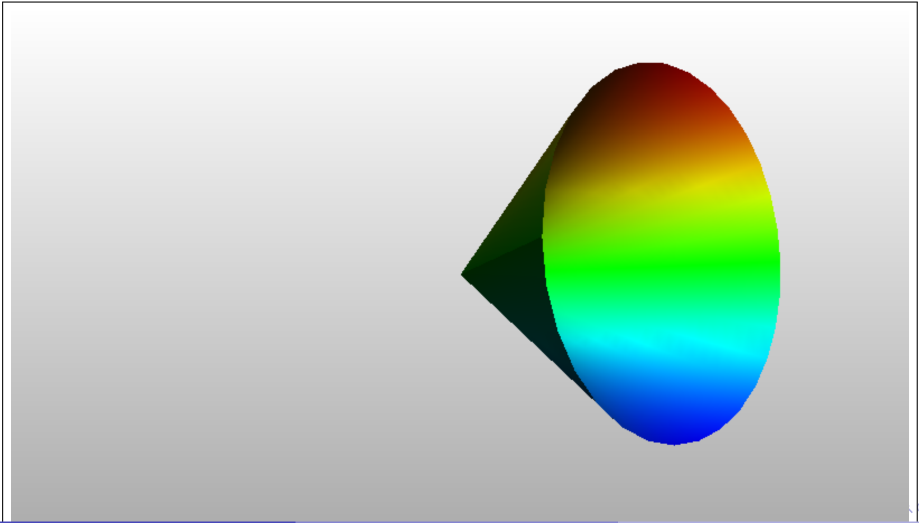

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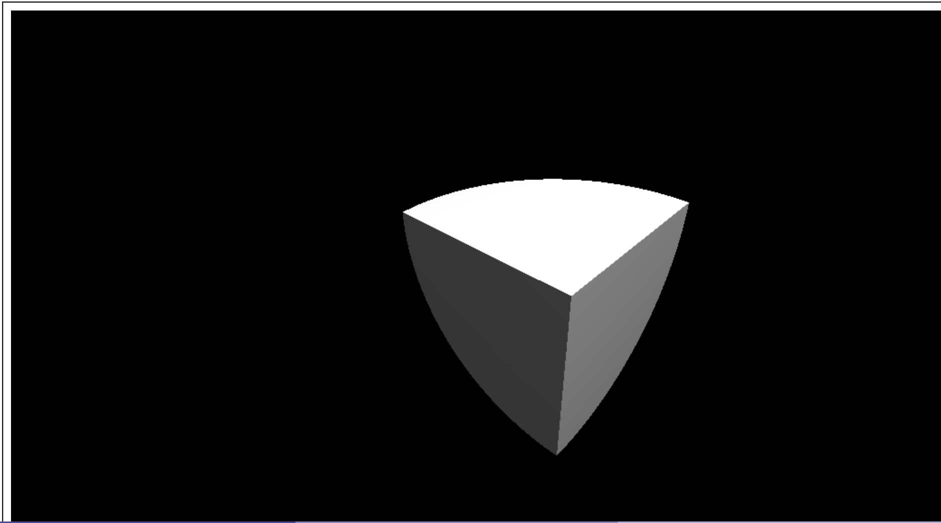
The geometry of the double solid cube is the following : at a vertex of the cube, we have to glue two cones over $1/8$ of sphere.



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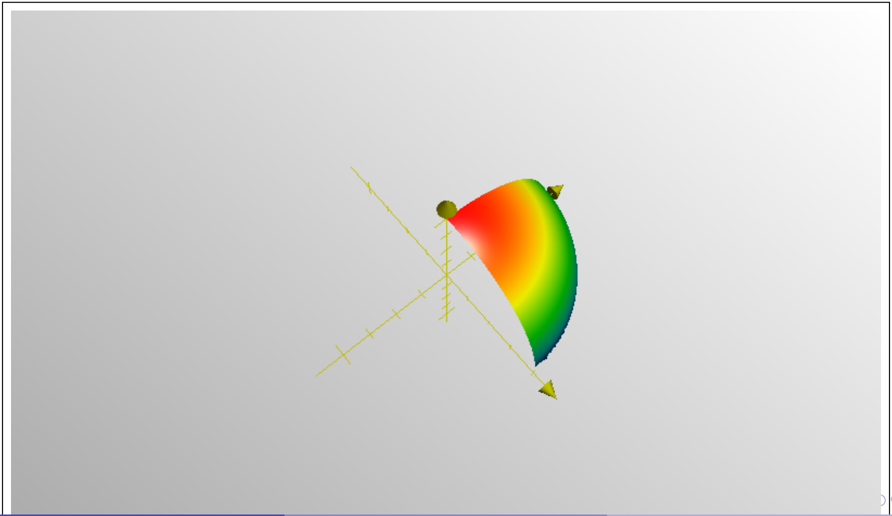

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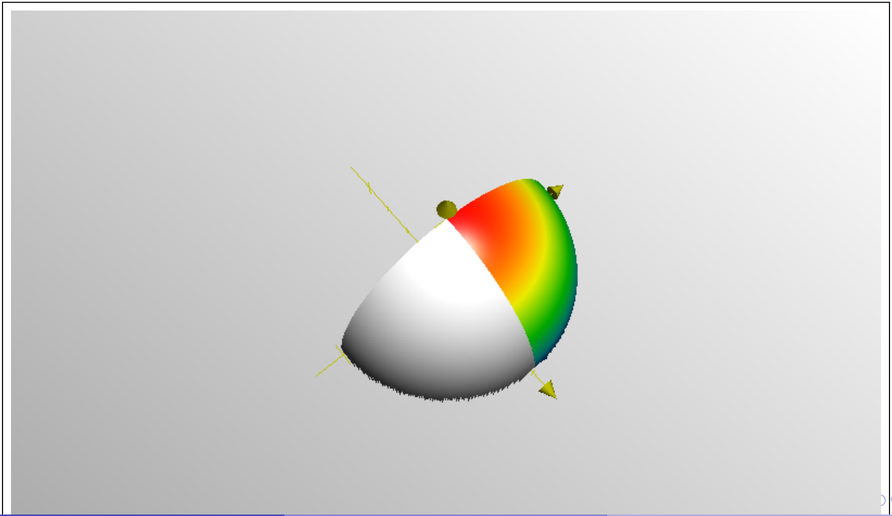

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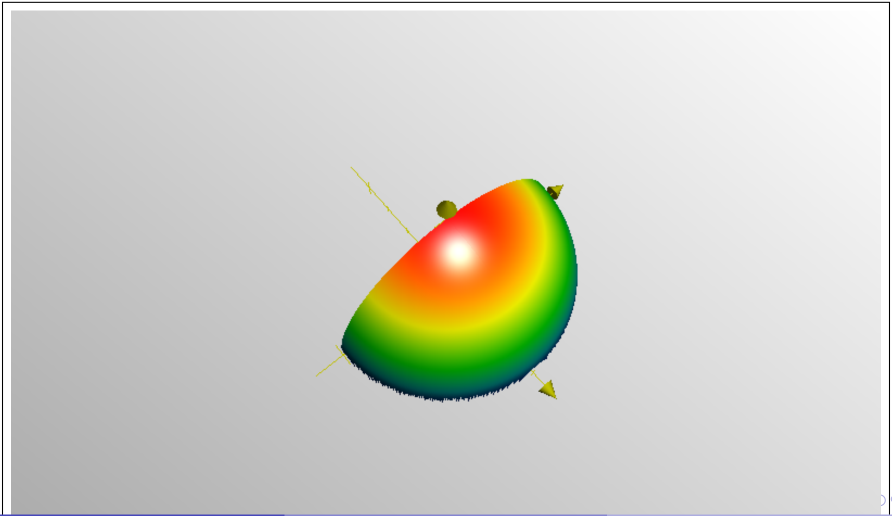

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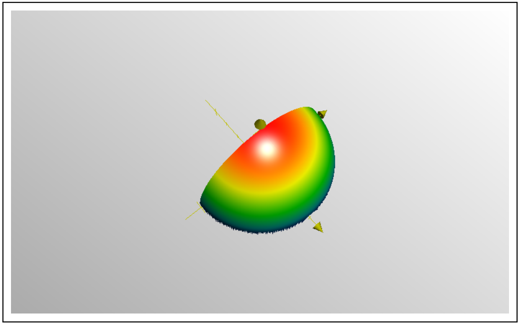
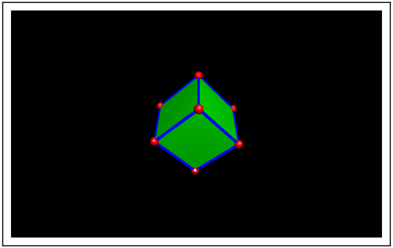


This is a cone over the berlingot !

Some stratified space : the double solid cube

Summary : The double solid cube has a decomposition $X \supset X_1 \subset X_0$:



- On $X \setminus X_1$ the geometry is Euclidean
- $X_1 \setminus X_0$ is the union of 12 unit segments and at a point on $X_1 \setminus X_0$, the geometry is the product of an interval with a cone whose link has length π.
- X_0 consists of 8 points and the geometry near these points looks like a cone over a berlingot.

What are stratified spaces ?

The basics object are cone over metric space : if Σ is a complete metric space with distance d_Σ, the cone $C(\Sigma)$ over Σ is the completion of the product $(0,\infty)\times\Sigma$ with the distance for
$p=(t,x),\ q=(s,y)\in(0,\infty)\times\Sigma$

$$d(p,q)=\begin{cases} t+s & \text{if} \quad d_Y(x,y)\geq\pi \\ \sqrt{t^2+s^2-2ts\cos d_Y(x,y)} & \text{if} \quad d_Y(x,y)\leq\pi \end{cases}$$

We have only to blown down $\{0\}\times\Sigma$ to a point (the vertex of the cone) from $[0+\infty)\times\Sigma$.

What are stratified spaces ?

A stratified space is a compact metric space (X, d) with a stratification

$$X \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0$$

such that

- near each point $x \in X \setminus X_{n-2} = X_{\mathrm{reg}}$, the geometry is Riemannian (and of dimension n).

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such that

- near each point $x \in X \setminus X_{n-2} = X_{\mathrm{reg}}$, the geometry is Riemannian (and of dimension n).
- near each point $x \in X_k \setminus X_{k-1}$, the geometry looks like a product

$$\mathbb{R}^k \times C(\Sigma_x)$$

where Σ_x is a $(n-k-1)$- dimensional stratified space.

The scalar curvature of stratified spaces with iterated edge metric

- The regular part $X_{reg} = X \setminus X_{n-2}$ is a smooth open Riemannian manifold and as $\mathrm{vol}\, X_{n-2} = 0$ we can extend $\mathrm{Scal}_g \colon X \to \mathbb{R}$ to a measurable function, we have $\mathrm{Scal}_g(x) = \mathcal{O}\left(1/\mathrm{dist}^2(x, X_{n-2})\right)$.

The scalar curvature of stratified spaces with iterated edge metric

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- As $\mathcal{C}_0^\infty(X_{reg})$ is dense in $H^1(X)$ we get in fact

$$Y(X) = \inf_{u \in H^1(X)} \left\{ \int_X \left[c_n |du|^2 + \mathrm{Scal}_g u^2 \right] \mathrm{dv}_g, \int_X |u|^{\frac{2n}{n-2}} \mathrm{dv}_g = 1 \right\}$$

where $c_n = 4(n-1)/(n-2)$.

Yamabe flow on stratified spaces

The Yamabe flow is a parabolic flow $g = u^{\frac{4}{n-2}} g_0$:

$$\frac{4}{n-2} \frac{\partial}{\partial t} u = \sigma(t) u - u^{-\frac{4}{n-2}} \left(\frac{4(n-1)}{n-2} \Delta_{g_0} u + \mathrm{Scal}_{g_0} u \right),$$

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Our convention for Δ_g is that

$$\int_M |du|_g^2 \mathrm{dv}_g = \int u\,\Delta_g u\;\mathrm{dv}_g.$$

Hence on $\mathbb{R}^n$: $\Delta_{\mathrm{eucl}} = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

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Theorem (Carron, Olsen Lye & Vertman, 2021)

Assume that X is a stratified space of dimension $n > 2$ and that g_0 is a Riemannian metric on X_{reg} such that

$$Y(X) > 0 \text{ and } \mathrm{vol}_{g_0} X_{reg} = 1 \text{ and } \mathrm{Scal}_{g_0} \in L^{p > \frac{n}{2}}$$

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This a parabolic counterpart of some existence result for the Yamabe problem on stratified spaces [Akutagawa-C-Mazzeo 2014].

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We used functional analytic method to get the existence of solution of equation of the type

$$\frac{\partial}{\partial t} u = u^{\beta} \left(-Lu + Vu\right)$$

on Dirichlet space $(X, d, \mu, \mathcal{E})$ where

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$$\forall\, w \in \mathcal{D}(\mathcal{E}) \colon\ \|w\|^2_{L^{\frac{2n}{n-2}}} \leq A \left(\mathcal{E}(w) + \|w\|^2_{L^2}\right).$$

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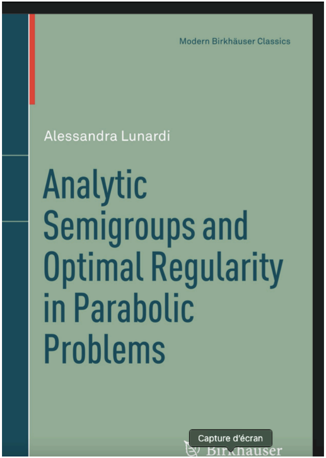
This abstract framework generalizes $\mathcal{E}(u) = \int_M |du|^2_g \mathrm{dv}_g = \int_M \Delta_g u u \mathrm{dv}_g$.

Yamabe flow on stratified spaces, existence

Together with the suitable general theorem that can be found in the very good book :



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Then there is a long time solution of the Yamabe flow starting at g_0.

This above general theorem also provides useful information about the regularity of the Scalar curvature.

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- But we know that $\Delta_g \mathrm{Scal}_g \in L^p$ ($\Rightarrow t > 0 : \mathrm{Scal}_g(t) \in L^\infty$).
- And we can deduce Scalar curvature estimate from the equations :

$$\frac{\partial}{\partial t} \mathrm{Scal}_{g(t)} + (n-1) \Delta_{g(t)} \mathrm{Scal}_{g(t)} = -\mathrm{Scal}_{g(t)} \left(\mathrm{Scal}_{g(t)} - \sigma(t) \right).$$

Yamabe flow on stratified spaces, asymptotic behavior

A brief summary about the large time behavior :

- We also have a dichotomy compactness vs concentration (formation of bubbles).

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- We also have a dichotomy compactness vs concentration (formation of bubbles).
- We do not have a positive mass theorem.
- There are examples where only concentration occurs.

Yamabe flow on stratified spaces, compactness vs concentration

Theorem

Let $(X^{n>2}, g_0)$ be a stratified space such that

$$Y(X) > 0 \text{ and } \mathrm{vol}_{g_0} X_{reg} = 1 \text{ and } \mathrm{Scal}_{g_0} \in L^{p>\frac{n}{2}},$$

and $t \mapsto u(t)$ be the solution of the Yamabe flow. There are $t_k \to \infty$ and find $u_\infty \in H^{2,p}(X)$ solving the Yamabe equation

$$c_n \Delta_0 u_\infty + \mathrm{Scal}_{g_0} u_\infty = \sigma_\infty u_\infty^{\frac{n+2}{n-2}}$$

Such that

- $u_k := u(t_k) \overset{H^1}{\rightharpoonup} u_\infty$,
- there is a finite set $F = \{x_1, \dots, x_L\} \subset X$ with $u_k \overset{C^\alpha_{loc}(X \setminus F)}{\longrightarrow} u_\infty$.

When $F = \emptyset$, we do have convergence of $u_k \overset{C^\alpha(X)}{\longrightarrow} u_\infty$.

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- In some case, we know that there is no Yamabe minimizers i.e. solution of

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- For instance : Akutagawa, Mondello (2019) branched cover of $\mathbb{S}^n$ over $\mathbb{S}^{n-2} \subset \mathbb{S}^n$ or S. Brendle (2020) for ALE manifolds with Ricci ≥ 0.

Thanks you for your attention !