Distribution L^2 Norms on Analytic Functions - the Drury-Arveson Space

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Observation

It all starts with a naive observation...

Recall: For a complex linear space X, an *inner product* is a sesquilinear form that

(a) $\langle x, y\rangle = \overline{\langle y, x\rangle}, \forall x, y \in X$;

(b) $\langle x, x\rangle \geq 0, \forall x \in X$;

(c) $\langle x, x\rangle = 0$ if and only if $x = 0$.

A rich source of inner products come from positive measures:

$$\langle f, g\rangle = \int f\bar{g}\mathrm{d}\mu.$$

(Let us not worry about condition (c) for the moment)

Observation (continued...)

Also Recall: A measure is a special class of distribution.
A *distribution* in $\mathbb{C}^n$ is an element in the dual of the test function space $\mathcal{D}(\mathbb{C}^n)$ (compactly supported smooth functions). A distribution can be a locally integrable function, a measure, or derivatives of them:

$$\partial^\alpha u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi), \quad \forall \phi \in \mathcal{D}(\mathbb{C}^n).$$

Observation: For a distribution u,

$$u(|\phi|^2) \geq 0, \forall \phi \in \mathcal{D} \quad \Leftrightarrow \quad u \text{ is a positive measure.}$$

However, the collection

$$\mathcal{U} := \{u \in \mathcal{D}^* : u(|\phi|^2) \geq 0, \forall \phi \in \mathbb{C}[z_1, \ldots, z_n]\}$$

contains more than positive measures.

Distribution L^2 Norms on Holomorphic Functions

Therefore we can talk about "L^2 norms defined by a distribution" on the space of functions. For any $u \in \mathcal{U}$ and f holomorphic in a neighborhood of $\mathrm{supp} u$, define

$$\|f\|_u^2 = u(|f|^2) := \int |f|^2 \mathrm{d}u.$$

Naturally, we can define an inner product by

$$\langle f, g \rangle_u = u(f\bar{g}) := \int f\bar{g} \mathrm{d}u.$$

Distribution L^2 Norm - Examples

In fact, we have already seen such norms...

- Choose a sufficiently small $\epsilon > 0$ and define

$$u = -\chi_{\epsilon\mathbb{B}_n} + \chi_{\mathbb{B}_n \setminus \epsilon\mathbb{B}_n}.$$

  Then $\|\cdot\|_u$ is equivalent to the Bergman norm on $\mathbb{B}_n$.
- Define $u = \Delta + \delta_0$. Then for $f \in \mathrm{Hol}(\mathbb{B}_n)$,

$$\|f\|_u^2 = \int_{\mathbb{B}_n} \Delta|f|^2(z)\mathrm{d}m(z) + |f(0)|^2 = \int_{\mathbb{B}_n} |\nabla f(z)|^2 \mathrm{d}m(z) + |f(0)|^2.$$

  So $\|\cdot\|_u$ is exactly the Dirichlet norm.
- The Besov-Sobolev norms can also be defined this way...(We will come back to this later.)

So are we reinventing the wheel or does this definition give something new?

The Drury-Arveson Space H_n^2

Let us take a look at the Drury-Arveson space.

The Drury-Arveson space H_n^2 is a Hilbert space of holomorphic functions on $\mathbb{B}_n$. There are several equivalent definitions:

- H_n^2 has orthogonal basis $\{z^\alpha\}_{\alpha\in\mathbb{N}_0^n}$.

$$\|z^\alpha\|_{H_n^2}^2 = \frac{\alpha!}{|\alpha|!}.$$

- H_n^2 has reproducing kernels

$$K_z(w) = \frac{1}{1-\langle w, z\rangle}, \quad \forall z, w \in \mathbb{B}_n.$$

- H_n^2 is the symmetric Fock space.

The Drury-Arveson Space H_n^2 (continued...)

- Up to an equivalent norm, H_n^2 is a Besov-Sobolev space: take an integer $m > \frac{n-1}{2}$, then $\| \cdot \|_{H_n^2} \approx ||| \cdot |||$, where

$$|||f|||^2 = \sum_{|\alpha|<m} \left| \frac{\partial^\alpha f}{\partial z^\alpha}(0) \right|^2 + \sum_{|\alpha|=m} \int_{\mathbb{B}_n} \left| \frac{\partial^\alpha f}{\partial z^\alpha}(z) \right|^2 (1-|z|^2)^{2m-d} \mathrm{d}m(z).$$

With the previous observation, we add another definition of the Drury-Arveson space...

The Drury-Arveson Space - a different definition

Define the "radial derivative" operator

$$R = \frac{1}{2}\left[z_1\partial_{z_1} + \cdots + z_n\partial_{z_n} + \bar{z_1}\bar{\partial}_{z_1} + \ldots + \bar{z_n}\bar{\partial}_{z_n}\right].$$

Denote $\mathrm{d}\sigma$ the normalized surface measure of $\mathbb{S}_n = \partial\mathbb{B}_n$.

Define a distribution ν supported on $\mathbb{S}_n$: for any function ϕ smooth in a neighborhood of $\mathbb{S}_n$,

$$\nu(\phi) = \int_{\mathbb{S}_n} \frac{1}{(n-1)!}\big(R + (n-1)I\big)\big(R + (n-2)I\big)\ldots\big(R + I\big)\phi \mathrm{d}\sigma.$$

In other words, ν is the following derivative of the surface measure (in the sense of distribution):

$$\nu = \frac{1}{(n-1)!}\big((n-1)I - R\big)\ldots\big(I - R\big)\sigma.$$

The $\|\cdot\|_\nu$ Norm Agrees with the DA Norm

Theorem

(1) *For* $f \in \mathrm{Hol}(\overline{\mathbb{B}_n})$*,*

$$\|f\|_{H_n^2}^2 = \nu(|f|^2).$$

(2) *In general, for* $f \in H_n^2$*,*

$$\|f\|_{H_n^2}^2 = \|f\|_\nu^2 := \sup_{0<r<1} \nu(|f_r|^2) = \lim_{r\to 1^-} \nu(|f_r|^2),$$

where $f_r(z) = f(rz)$*.*

(3) *The Drury-Arveson space can be defined as*

$$H_n^2 = \{f \in \mathrm{Hol}(\mathbb{B}_n) : \sup_{0<r<1} \nu(|f_r|^2) < \infty\}.$$

The proof is straightforward, after noticing that $R(z^\alpha \bar{z}^\beta) = \frac{|\alpha|+|\beta|}{2} z^\alpha \bar{z}^\beta$.

Some Further Thoughts - Explicit Formula

Recall that the Drury-Arveson norm is equivalent to the Besov-Sobolev norm $|||\cdot|||$. With the distribution representation and some integration by parts we can write the DA norm in Besov-Sobolev style.

For example, if $n = 2$, then

$$\|f\|_{H_n^2}^2 = |f(0)|^2 + \pi^{-2} \int_{\mathbb{B}_2} \frac{1 - \ln |z|^2}{|z|^4} |Nf(z)|^2 \mathrm{d}m(z).$$

Here $N = z_1 \partial_{z_1} + \cdots + z_n \partial_{z_n}$.

The explicit formulas for other dimensions n can also be computed.

Some Further Thoughts - Justifying the Name Drury-Arveson Hardy Space

The $\|\cdot\|_\nu$ definition justifies the name "Drury-Arveson Hardy space" in that ν is supported on the sphere, and the $\|\cdot\|_\nu$ norm is defined as a limit.

To make it more like the Hardy space, we can work on the following problem:
Let us formally denote $\binom{R+n-1}{n-1} = \frac{1}{(n-1)!}\big(R+(n-1)I\big)\dots\big(R+I\big)$.
For any $\zeta \in \mathbb{S}_n$ and $f \in \mathrm{Hol}(\overline{\mathbb{B}_n})$, we have the vector $(\partial^\alpha f(\zeta))_{|\alpha|\le n-1}$.
Then the operation $(\partial^\alpha f(\zeta))_{|\alpha|\le n-1} \mapsto \binom{R+n-1}{n-1}|f|^2(\zeta)$ extends to a Hermitian form.
Recall that $\|f\|_\nu^2 = \int_{\mathbb{S}_n} \binom{R+n-1}{n-1}|f|^2(\zeta)\mathrm{d}\sigma(\zeta)$. Thus we can now embed H_n^2 into a subspace of the "L^2 sections" of some vector bundle on $\mathbb{S}_n$.
This resembles the two definitions of the Hardy space: $H^2(\mathbb{B}_n)$ and $H^2(\mathbb{S}_n)$.
Thus it makes sense to look for an analogue of the K-limit (non-tangential limit), which would make the correspondence more explicit.

Some Further Thoughts - Multipliers

Recall that the *Multiplier algebra* $\mathcal{M}_n$ of H_n^2 is

$$\mathcal{M}_n := \mathrm{Mult}(H_n^2) = \{f : \mathbb{B}_n \to \mathbb{C} | fh \in H_n^2 \text{ for all } h \in H_n^2\}.$$

Some properties of $\mathcal{M}_n$:

- (Arveson) $\mathcal{M}_n \subsetneq H^\infty(\mathbb{B}_n)$;
- (Arveson) the ball algebra $A(\mathbb{B}_n) \not\subset \mathcal{M}_n$;
- (Fang and Xia) $f \in H_n^2, \sup_{z \in \mathbb{B}_n} \|fk_z\| < \infty \not\Rightarrow f \in \mathcal{M}_n$.

Some Further Thoughts - Multipliers

A successful characterization of functions in $\mathcal{M}_n$ is by Ortega, Fàbrega, and Arcozzi, Rochberg, Sawyer:

- (Ortega & Fàbrega) Let $f \in H^\infty(\mathbb{B}_n)$, and $m > \frac{n-1}{2}$. Then $f \in \mathcal{M}_n$ if and only if the measure

$$\mathrm{d}\mu_{f,m} = \sum_{|\alpha|=m} |\frac{\partial^\alpha f}{\partial z^\alpha}(z)|^2(1-|z|^2)^{2m-n}\mathrm{d}m(z)$$

  is a Carleson measure for H^2_n.

- (Arcozzi, Rochberg & Sawyer) A positive measure μ on $\mathbb{B}_n$ is a Carleson measure if and only if it satisfies both the "simple condition" and the "split tree condition". The conditions are based on their Bergman tree construction.

Some Further Thoughts - Multipliers

Continuing with the previous thought: for any $f \in \mathrm{Hol}(\overline{\mathbb{B}_n})$ corresponds its "boundary section" $\left(\frac{\partial^\alpha f}{\partial z^\alpha}(\zeta)\right)_{|\alpha| \leq n-1}$, $\zeta \in \mathbb{S}_n$.

For any $g \in \mathcal{M}_n$ and $z \in \mathbb{B}_n$, the mapping

$$\left(\frac{\partial^\alpha f}{\partial z^\alpha}(z)\right) \mapsto \left(\frac{\partial^\alpha gf}{\partial z^\alpha}(z)\right)$$

determines a linear transformation.

Question: is there a characterization of $g \in \mathcal{M}_n$ in terms of these linear transformations?

Some Further Thoughts - Toeplitz Operators on H_n^2

A multiplier $g \in \mathcal{M}_n$ defines a multiplication operator

$$M_g f = gf, \quad \forall f \in H_n^2.$$

In particular we have the multiplication operators of coordinate functions: $M_{z_i}, i = 1, \ldots, n$.

The *Toeplitz algebra* $\mathcal{T}_n$ is the C^*-algebra generated by $\{I, M_{z_1}, \ldots, M_{z_n}\}$.

Arveson proved that

$$\mathcal{T}_n/\mathcal{K} \cong C(\mathbb{S}_n).$$

Some Further Thoughts - Toeplitz Operators on H^2_n

In the Toeplitz algebra $\mathcal{T}_n$ of H^2_n, the only explicitly defined operators are M_{z_i}, $M^*_{z_i}$, $i = 1, \ldots, n$, and the combinations of them.

Recall the in the Bergman space version of the Toeplitz algebra, more operators are defined: for a "symbol" $g \in L^\infty(\mathbb{B}_n)$, we can define the Toeplitz operator

$$T_g f(z) = \int_{\mathbb{B}_n} g(w) f(w) \overline{K_z(w)} \mathrm{d}m(w).$$

With the distribution ν we are allowed to define the following "Toeplitz operators on H^2_n": for a "symbol function" g that is $\mathcal{C}^{n-1}$ in a neighborhood of $\mathbb{S}_n$, define

$$T_g f(z) = \nu(g f \overline{K_z}), \quad \forall f \in \mathrm{Hol}(\overline{\mathbb{B}_n}), \forall z \in \mathbb{B}_n.$$

Some Further Thoughts - Toeplitz Operators on H_n^2

It is easy to see that:

- $T_g = M_g, \quad \forall g \in \mathcal{M}_n;$
- $T_{\bar{g}} = M_g^*, \quad \forall g \in \mathcal{M}_n;$
- $T_{\bar{g}_1 g_2} = M_{g_2}^* M_{g_1}, \quad \forall g_1, g_2 \in \mathcal{M}_n.$

Thus we can ask a lot of classical questions:

- When is T_g bounded, compact, Schatten class, positive, etc.?
- Properties of commutators
- Spectrum, trace, etc.

Some Further Thoughts - von Neuman/Drury Inequality

A *contraction* is a bounded operator T on a Hilbert space $\mathcal{H}$ such that $\|T\| \leq 1$. It is well-known that

(von Neumann's inequality) For every contraction T and every polynomial p,

$$\|p(T)\| \leq \sum_{|z| \leq 1} |p(z)|.$$

A *n-contraction* is a commuting tuple $(T_1, \ldots, T_n)$ of operators on $\mathcal{H}$ such that $\sum_{i=1}^{n} T_i T_i^* \leq I$. We also have the famous Drury's inequality

(Drury's Inequality) Let $(T_1, \ldots, T_n)$ be a n-contraction. Then for every polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$,

$$\|p(T)\| \leq \|M_p\|_{\mathcal{M}_n}.$$

Some Further Thoughts - von Neumann/Drury Inequality

Both inequalities are obtained by dilating the operator (tuple of operators) into some model space, and these model spaces are all tensors of function spaces.

Meanwhile, for commuting tuples of contractions, things are complicated. In particular, Varopoulos gives a commuting triple (R, S, T) and a polynomial $p \in \mathbb{C}[z_1, z_2, z_3]$ such that

$$\|p(R, S, T)\| > \sup_{|x|,|y|,|z|\leq 1} |p(x, y, z)|.$$

Now that we know the Drury-Arveson space is defined by a distribution, it makes sense to ask:

Question: Is there a distribution μ supported on $\overline{\mathbb{D}^n}$ such that $\|\cdot\|_\mu$ defines an analytic function space that serves as a model space for commuting contractions?

Other Questions

Other interesting questions...

- distribution L^p norms on analytic function spaces?
- Carleson distributions?

Thank you!