Hardy Sobolev spaces in several complex variables

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The classical **Hardy space** $H^2(\mathbb{B}^d)$ is defined as the space of holomorphic functions $f \in \mathcal{O}(\mathbb{B}^d)$ such that

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_{\partial\mathbb{B}^d} |f(r\zeta)|^2 d\sigma(\zeta) < +\infty.$$

Let also $\mathcal{R}^s$ be the **fractional differentiation operator**

$$\mathcal{R}^s\Big(\sum_{\alpha\in\mathbb{N}^d} c_\alpha z^\alpha\Big) := \sum_{\alpha\in\mathbb{N}^d} (|\alpha|+1)^s c_\alpha z^\alpha.$$

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We define the **Hardy-Sobolev space** H^2_s as the space of $f \in \mathcal{O}(\mathbb{B}^d)$ such that

$$\|f\|_s := \|\mathcal{R}^s f\|_{H^2} < +\infty.$$

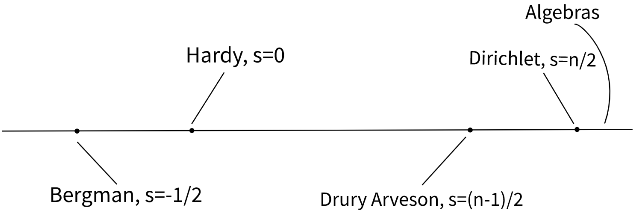

Figure: Scale of H^2_s spaces

$$K_s(z,w) = \frac{1}{(1 - z\cdot\overline{w})^{d-2s}}, s < d/2; \quad K_{\frac{d}{2}}(z,w) = \frac{1}{z\cdot\overline{w}} \log \frac{1}{1 - z\cdot\overline{w}}.$$

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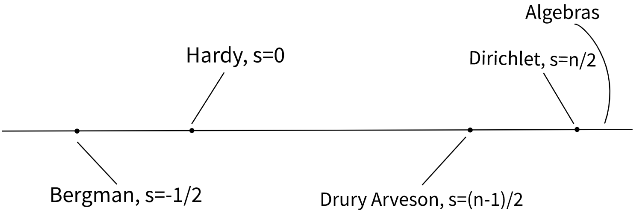

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But why ?

For *operator theorists* Drury - Arveson's space is of fundamental importance, essentially because of Drury's inequality;

Theorem (Drury's von Neumann type inequality)

Let $A_1, \dots A_d$ a commuting row of operators on a Hilbert space $\mathcal{H}$ such that

$$\sum_{i=1}^{d} A_i^* A_i \leq \mathrm{id}\,.$$

Then for any complex polynomial p of d variables we have

$$\|p(A_1, \dots, A_d)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\|f\| \leq 1} \|pf\|.$$

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From a *geometric* viewpoint the Dirichlet space is particularly interesting.

Theorem (Arazy & Fisher (1985) $d = 1$, Peloso (1992) $d > 2$)

The Dirichlet space ($s = \frac{d}{2}$) is the "unique" Hilbert space of analytic functions in the unit ball which contains constants and is invariant under composition with biholomorphisms of the unit ball.

In fact there exists seminorms for $H^2_{\frac{d}{2}}$ such that $\|f \circ \varphi\| = \|f\|, \forall \varphi \in Aut(\mathbb{B}^d)$. For $d = 1$ this seminorm is exactly the square root of the area of $f(\mathbb{B}^1)$.

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Let $\mathcal{M}(H_s^2)$ be the space of functions f in the unit ball such that

$$f \cdot g \in H_s^2, \quad \forall g \in H_s^2.$$

This is an **Banach algebra** equipped with the norm of the multiplication operator, i.e.;

$$\|f\|_{\mathcal{M}(H_s^2)} := \sup_{\|g\|_{H_s^2} \leq 1} \|f \cdot g\|_{H_s^2}.$$

(Recall Drury's inequality) It can be proven that

$$\|f\|_{\mathcal{M}(H_s^2)} \approx \|f\|_{H^\infty} + [f]_{CM,s}$$

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To understand the quantity $[f]_{CM,s}$ we need to introduce **Carleson measures**. Let μ a positive Borel measure on $\mathbb{B}^d$.

We say that μ is a Carleson measure for H^2_s if $H^2_s \subseteq L^2(\mathbb{B}^d, d\mu)$.

The **Carleson constant** of μ is the norm of the identity operator $\mathrm{id} : H^2_s \to L^2(\mathbb{B}^d, d\mu)$.

Then $[f]_{CM,s}$ is the Carleson constant of the positive Borel measure,

$$|(1-|z|)^m \partial^m f(z)|^2 (1-|z|)^{d-2s} d\lambda_d(z).$$

Where $m > s$ is an integer and the quantity is comparable for all $m > s$.

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Let us introduce a type of **capacity** for sets in $\partial\mathbb{B}^d$, $\frac{d}{2} \geq s > 0$.

- The s-**potential** of μ is

$$\mathcal{I}_{2s}(\mu)(z) := \int_{\partial\mathbb{B}^d} |K_s(z,w)|\, d\mu(w).$$

- The **s-energy** of μ is defined by

$$\mathcal{E}_s(\mu) = \int_{\partial\mathbb{B}^d} \int_{\partial\mathbb{B}^d} |K_s(z,w)| d\mu(z) d\mu(w).$$

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Theorem (Stegenga 1980, Ahern & Cohn 1989)

Let $\frac{d}{2} \geq s > \frac{d-1}{2}$, then a (positive Borel) measure μ is Carleson for H^2_s if and only if for all $\zeta_1, \dots \zeta_k \in \partial\mathbb{B}^d, r_1, \dots r_k < 1$ we have

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On a macroscopic scale the flow of the argument for sufficiency is the following.

(1) The kernel defining the s-potential is the reciprocal of a quasidistance, i.e.

$$\frac{1}{|K_s(z,w)|} \leq C\Big(\frac{1}{|K_s(z,y)|} + \frac{1}{|K_s(y,w)|}\Big), \quad z, y, w \in \partial\mathbb{B}^d.$$

(2) This implies (Adams & Hedberg) that the potential satisfies the so called *boundedness principle*, i.e.

$$\|\mathcal{I}_{2s}(\mu)\|_{L^\infty(\partial\mathbb{B}^d)} \leq M\|\mathcal{I}_{2s}(\mu)\|_{L^\infty(\operatorname{supp}\mu)}$$

(3) In turn this implies a *strong capacitary inequality*;

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Then the Carleson inequality as follows. Pick a function $F \in H_s^2$. The real part $f := ReF$ has a representation as $f = \mathcal{I}_s(\varphi)$ such that $\|\varphi\|_{L^2(d\sigma)} \lesssim \|F\|_{H_s^2}$.

$$\begin{aligned}\int_{\mathbb{B}^d} |f|^2 d\mu &\leq \int_0^\infty \mu(\mathcal{I}_s(|\varphi|) > \lambda) d\lambda^2 \\ &\lesssim \int_0^\infty C_s(\mathcal{I}_s(|\varphi|) > \lambda)) d\lambda^2 \\ &\lesssim \|\varphi\|_{L^2(d\sigma)}^2 \lesssim \|F\|_{H_s^2}^2.\end{aligned}$$

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So what goes wrong for $s \leq \frac{d-1}{2}$. Nothing... only that considering a potential associated to the **absolute value** of the kernel K_s is too crude in most cases.

It is only the **real part of the kernel** that only matters. It just happens that for $\frac{d}{2} \geq s > \frac{d-1}{2}$ we have

$$ReK_s(z,w) \approx |K_s(z,w)|, \quad z,w \in \mathbb{B}^d.$$

For the Drury Arveson space $s = \frac{d-1}{2}$ the real part of the kernel is still positive, while for $s < \frac{d-1}{2}$ the real part of the kernel is signed (things are even worse in some sense).

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By **testing** $\Theta^*\Theta : L^2(d\mu) \to L^2(\mu)$ on characteristic functions of sets of the form $Q_r(\zeta)$ we have the following necessary condition for the Carleson measure μ;

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The general idea of interpolation problems is that one is asked to construct (or prove the existence) of a function **in some admissible space** which in some set of points assumes preassigned values.

For example the elementary fact that for any complex numbers $z_1, z_2, \dots z_n, w_1, \dots w_n$ there exists **a polynomial** p **of degree less than** n such that $p(z_i) = w_i$, is a an interpolation result.

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We would like to study interpolation problems that the space of admissible functions consists of **holomorphic functions** and carries some **Hilbert space structure.**

Let $\mathcal{H}$ a rkHs in the unit disc and $\mathcal{Z} := \{z_i\} \subseteq \mathbb{B}^d$ a sequence, the associated weighted restriction operator are defined as follows.

$$T_{\mathcal{Z}} : \mathcal{H} \dashrightarrow \ell^2$$
$$f \mapsto \left\{ \frac{f(z_i)}{\|K_{z_i}\|} \right\}$$

The dashed arrow means that a priori $T_{\mathcal{Z}}$ is not defined everywhere. If $T_{\mathcal{Z}}$ is surjective we say that it is **simply interpolating** (SI) (also onto interpolating exists in the literature). Explicitly

$$\forall \{a_i\} \in \ell^2 \; \exists f \in \mathcal{H} \text{ such that } f(z_i) = a_i \|K_{z_i}\|.$$

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1. The boundedness of $T_{\mathcal{Z}}$ is equivalent to say that the measure

$$d\mu_{\mathcal{Z}} := \sum_{z \in \mathcal{Z}} \frac{\delta_z}{\|K_z\|^2}$$

is **Carleson** for $\mathcal{H}$, i.e. $\mathcal{H} \subseteq L^2(\mathbb{B}^d, d\mu_{\mathcal{Z}})$.

2. A geometric condition which is implied by simple interpolation is the so called **weak separation** (WS). This can be expressed in terms of the **Gleason metric**

$$d_G(z, w) := \sqrt{1 - \frac{|\langle K_z, K_w \rangle|^2}{\|K_z\|^2 \|K_w\|^2}} = |\sin \angle (K_z, K_w)|.$$

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2. For Hardy Sobolev spaces $H^2_s, s < d/2$ weak separation is equivalent to separation with respect to the **Bergman metric in the unit ball**. For $s = \frac{d}{2}$ the weak separation condition is more complicated.

Theorem

Let $\frac{d-1}{2} \leq s \leq \frac{d}{2}$. Then a sequence $\mathcal{Z} \subseteq \mathbb{B}^d$ is universally interpolating for H^2_s if and only if it is weakly separated and $d\mu_{\mathcal{Z}}$ is a Carleson measure.

- For $d = 1, s = 0$ Carleson 1958, Shapiro & Shields 1961
- For $d = 1, 0 < s \leq \frac{1}{2}$ Bishop 1994 (preprint), Marshall and Sundberg 1994 (preprint)
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In some way random sequences give us a sense of which situations are "generic". One possible way to consider random sequences are the so called Steinhaus sequences. Let ζ_n an independent random sequence of points in $\partial\mathbb{B}^d$ distributed according to the Lebesgue measure $d\sigma$ and a (deterministic) sequence of radii $\{r_n\} \subseteq [0, 1)$. Then the sequence $\Lambda = \{\Lambda_n\}$ of random variables

$$\Lambda_n = r_n \zeta_n$$

is called **Steinhaus sequence.** Notice that being interpolating (in any sense) is a **tail** event. Therefore Kolmogorov 0-1 theorem applies.
Hence there exists a condition on r_n which determines whether Λ_n is interpolating with probability 0 or 1. Same applies for weak separation, and the Carleson condition on $d\mu_{\mathcal{Z}}$.

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We introduce a counting function in order to fomulate our results;

$$N_n := \#\{r_i : n \leq \beta(0, r_i) < n+1\}$$

Theorem (C., Hartman, Kellay, Wick, 2021)

Let $d = 1,\ 0 < s < 1/4$, *then*

$$\mathbb{P}(\Lambda \ \textit{is UI for}\ H^2_s) = \begin{cases} 1, \\ 0 \end{cases} \quad \textit{iff} \quad \begin{cases} \sum_{n\geq 1} 2^{-n} N_n^2 < \infty \\ \sum_{n\geq 1} 2^{-n} N_n^2 = \infty. \end{cases}.$$

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For Hardy Sobolev spaces in higher dimensions similar results have been investigated by Dayan Wick and Wu.

Theorem (Dayan, Wick & Wu, 2018)

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Let $d \geq 2$ and $\frac{d-1}{2} \leq s < \frac{d}{2}$;

$$\mathbb{P}(\Lambda \textit{ is UI for } H^2_s) = \begin{cases} 1, \\ 0 \end{cases} \quad \textit{if} \quad \begin{cases} \sum_{n\geq 1} 2^{-n(d-2s)} N_n < \infty \\ \sum_{n\geq 1} 2^{-n(d-2s)} N_n = \infty. \end{cases}$$

Theorem (CHKW)

$$\mathbb{P}(\Lambda \text{ is UI for } H^2_{\frac{1}{2}}) = \begin{cases} 1, & \text{if } \sum_{n\geq 1} \frac{N_n}{n} < \infty \\ 0 & \sum_{n\geq 1} \frac{N_n}{n} = \infty. \end{cases}$$

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Thank you for your attention !