TIME-LIKE BOUNDARIES IN GENERAL RELATIVISTIC EVOLUTION PROBLEMS 19w5140

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1 Overview of the Field

The workshop was concerned, in broad terms, with geometric and PDE problems for Einstein's field equations in which time-like boundaries play a critical role. The particular interest in such situations is illustrated by the following examples. (I) In astrophysics time-like boundaries mark the transition between massive stars and their outer gravitational fields. Such transition must be dealt with appropriately to accurately capture, in particular, tidal deformations which, in turn, carry crucial information on the stars' equation of state. (II) In numerical general relativistic calculations, needed, for instance, to understand the content of gravitational radiation in recently detected signals, involve computational boundaries. Such boundaries must be handled so as to ensure stability of the solution and not negatively influence the physical information sought. (III) In the case of anti-de Sitter (AdS) type space-times, which raised so much interest recently, not least because of the AdS/CFT correspondence, they serve to characterize the specific decay properties of space-times at space-like and null infinity. This correspondence maps standard field theory questions (e.g. the description of superconductors, superfluids, quark-gluon plasmas) into geometrical ones. Since the field theory is defined over the AdS boundary metric, its asymptotic structure in general and its boundary in particular require an accurate description.

In the context of Einstein's field equations these situations give rise to quite different PDE problems. In the transition case, the boundary is defined by the joint evolution of two or more adjacent space-time regions covering respectively different matter models. In such free boundary value problems only the initial situation can and must be prescribed. In problems related to numerical relativity calculations and the computation of gravitational radiation, one is dealing with initial-boundary value problems (IBVPs) where initial as well as boundary conditions can and must be specified. In both these cases the boundaries are assumed to be at finite space-time locations. The AdS case is distinguished from them by an ideal boundary at space-like and null infinity which can be thought of as a limit of families of time-like hypersurfaces. Again initial and boundary conditions can and must be specified. The conditions which can be given depend on the space-time geometry and the choice of matter models.

While technically each of these settings requires its own particular investigation, they have in common the need for a detailed analysis of the Einstein equations in a neighborhood of a time-like boundary. This must go beyond the considerations known from the usual discussions of IBVPs for quasi-linear hyperbolic systems (see, e.g. [4, 29]). Due to the geometric nature of the equations there arises the necessity to control the preservation of constraints and gauge conditions during the evolutions and the need to ensure the covariance of the formulation, tasks which are much more difficult for IBVPs than for the standard Cauchy problem. Moreover, the notion of stability leads in the presence of time-like boundaries to questions not encountered before. Answers to these questions will have profound consequences for various other investigations of gravitational fields.

2 Recent Developments and Open Problems

The three types of problems indicated above have been of interest in general relativity (GR) for a long time. They are clearly important for working out the content and implications of the theory. However, because of their inherent mathematical difficulties, progress has been very slow with some of them. Understanding the IBVP is certainly a key step in understanding the technical aspects of the problems. The free boundary value problem still provides difficulties even in the Newtonian setting. Recently, some progress has been achieved here (see for example [32, 2, 40] and references given there) but a lot remains to be done before freely moving massive bodies with vacuum exterior (say) can be discussed in sufficient generality in terms of analytic methods.

On the other hand, in recent years there have been carried out various investigations in numerical GR which led to a large body of specialized results that are related in one way or the other to the IBVP (see the survey article [45], which also provides a long list of recent references). To draw general conclusion from these special results, a general theory of the IBVP would be needed as a basic requisite. First steps into this direction have been taken in [17, 30, 31], where the existence of well posed IBVPs for Einstein's vacuum field equations (with boundaries at finite locations) has been shown. However, because of quite unexpected difficulties an important question remains unanswered: it is not clear yet whether such problems can be formulated under general assumptions in a covariant way [15]. This raises important questions concerning the geometric and physical nature of the settings, the general applicability of the results, and the long time evolution of solutions controlled by boundary conditions. A sufficiently general, broadly applicable, covariant theory is thus still not available. Moreover, there still needs to be explored the richness of possible boundary conditions and data and their physical and geometric interpretation. The results of the various specialized numerical studies can be expected to shed a new light into these questions.

Regarding AdS-type solutions to Einstein's field equations, they have met with considerable interest of analysts in recent years (see [1, 23, 24, 25, 26, 13, 46] for recent articles with the kind of analysis we are interested in here). It turns out that in four spacetime dimensions the IBVP with the boundary at infinity can be transformed, under suitable assumptions on the asymptotic behavior at space-like and null infinity, into an IBVP with a time-like boundary at finite location. Remarkably, it admits a formulation that is covariant as well as well-posed [14]. The freedom to prescribe boundary conditions/data has been clarified and local in time existence of AdS-type solution is unproblematic.

Some of the articles mentioned above are concerned with the global structure of AdS-type solutions. However, the central question of whether the AdS solution is globally non-linearly stable (a problem that is quite different from the corresponding problems for the de Sitter and the Minkowski solution) has remained untouched for a long time. Recently, the AdS-stability problem has been explored for the first time with numerical methods. Assuming spherical symmetry and reflecting boundary conditions to make the situation numerically manageable, it has been shown that the Einstein-scalar field system with negative cosmological constant tends to develop black holes for data arbitrarily close to AdS data [7]. Interestingly, there is also considerable evidence that the setting admits "islands of stability". These results sparked quite a series of related numerical and simplified analytic studies (in particular, see the recent work in [36] which provides a proof for the instability of AdS for the Einstein-null dust system with an inner mirror, and [37] for a generalization to the case where the null dust is replaced with a collisionless kinetic gas of massless particles and the inner mirror is removed). Moreover, they raise some extremely interesting questions about the notion of stability in the AdS-framework in general, about the possible choices and the interpretation of boundary conditions and data, and about the effect the various boundary conditions may have on the long term evolution of space-times [16]. A detailed analysis of these problems would not only shed a new light on the IBVP for Einstein's field equations, but should also prove useful for a deeper understanding of the evolution properties of Einstein's field equations and the understanding of various theoretical aspects of Einstein's theory in general. Additionally, this behavior can have profound implications on thermalization questions within the AdS/CFT paradigm.

The problem of stability of AdS spacetime belongs to a broader context of recent studies of long-time behavior of nonlinear dispersive wave in the presence of geometric confinement. The goal of these studies is to understand how the energy injected to the system gets distributed over the degrees of freedom during the evolution, in particular whether energy can flow to arbitrarily small spatial scales. The past few years have witnessed a significant progress in understanding this issue in the context of nonlinear Schrödinger equations (and their variants) on tori [12, 21, 20, 18] but, to the best of our knowledge, very little is known in this respect for hyperbolic wave equations such as Klein-Gordon, Yang-Mills, wave maps, or Einstein's equations. Interestingly, it has been shown recently that the effective dynamical system describing the dynamics of asymptotically AdS spacetimes is structurally very similar to the effective system describing the dynamics of the Bose-Einstein condensate in a harmonic trap [6]. This intriguing parallel can be exploited to make analytic progress for both systems, and shed new light on such studies.

3 Presentation Highlights

During the workshop, talks were presented covering the three types of problems mentioned in the overview section. Below, we review the highlights in each of these problems, as well as some important progress in related topics.

3.1 Free boundary value problem

In his talk, Tetu Makino presented the following two results: (i) the existence of spherically symmetric equilibrium solutions of the Einstein-Euler-de Sitter equations and the behavior of small dynamical perturbations thereof, including the matching of the metric to the exterior (vacuum) Schwarzschild-de-Sitter metric [33]. This provides a solution to the free-boundary value problem in the spherically symmetric case. (ii) The existence of axially symmetric stationary metrics describing a slowly rotating mass under a weak gravitational field in a bounded region containing the support of the density [34]. Furthermore, some speculations regarding the problem of matching an axial symmetric stationary interior configuration to an asymptotically flat vacuum exterior were discussed (including the source problem for the Kerr metric). This has sparked many interesting discussions and motivated new research regarding the existence of time-periodic solutions and the problem of obtaining global axisymmetric spacetimes of the Euler-Einstein system.

Todd Oliynyk discussed his new approach to establishing the (local in time) well-posedness of the relativistic Euler equations for liquid bodies in vacuum [40]. The approach is based on a wave formulation of the relativistic Euler equations that consists of a system of nonlinear wave equations in divergence form together with a combination of acoustic and Dirichlet boundary conditions. The equations and boundary conditions of the wave formulation differ from the standard ones by terms proportional to certain constraints, and one of the main technical problems to overcome is to show that these constraints propagate, which is necessary to ensure that solutions of the wave formulation determine solutions to the Euler equations with vacuum boundary conditions. During his talk, Oliynyk described the derivation of the wave equation and boundary conditions, the origin of the constraints, how one shows that the constraints propagate, and energy estimates obtained from his new formulation involving the acoustic boundary conditions. These results are remarkable, since they present a solution of the free boundary value problem for a liquid body in vacuum without symmetry assumptions.

3.2 Artificial time-like boundary problem

Oscar Reula discussed the role played by the boundary conditions in hyperbolic evolution problems with constraints. Very often in physics the evolution systems one has to deal with are not purely hyperbolic, but contain also constraints and gauge freedom. After fixing the gauge freedom one obtain a new system with constraints which one would like to solve subject to initial and boundary values. In particular, these values have to imply the correct propagation of constraints. In general, after fixing some reduction to a purely evolutionary system, this is asserted by computing by hand what is called the constraint subsidiary system, namely a system which is satisfied by the constraints quantities when the fields satisfy the reduced evolution system. If the subsidiary system is also hyperbolic then for the initial data case the situation is clear: one needs to impose the constraints on the initial data and then they will correctly propagated along evolution. For the boundary data one needs to impose the constraints for all incoming constraint modes. This must be done by fixing some of the otherwise free boundary data, that is the incoming modes. Thus, there must be a relation between some of the incoming modes of the evolution system and all the incoming modes of the constraint subsidiary system. Under certain conditions on the constraints this relation is known and understood, but those conditions are very restrictive. In his talk, Reula reviewed the known results and discussed what is known so far for the general case and what are the open questions that still need to be addressed.

Federico Carrasco addressed the IBVP for neutron star magnetospheres. Force-free electrodynamics (FFE) describes a particular regime of magnetically dominated relativistic plasmas, which arise in several astrophysical scenarios of interest such as pulsars or active galactic nuclei. In those regimes, the electromagnetic field obeys a modified nonlinear version of Maxwell equations, while the plasma only accommodates to locally cancel out the Lorentz force. In his talk, Carrasco discussed the IBVP of FFE at some given astrophysical settings. In particular, it was shown that, when restricted to the correct constraint submanifold, the system is symmetric hyperbolic [9], from which appropriate boundary conditions are constructed [10, 11]. In particular, the focus was given on the treatment to mimic the perfectly conducting surface of a neutron star, where incoming and outgoing physical modes need to be combined in a very precise way. Furthermore, results from 3D simulations based on this approach were shown.

Olivier Sarbach provided a review of the well-posed initial boundary value problem for the vacuum Einstein field equations in harmonic coordinates [31]. After describing the harmonic formulation that was used, including the choice for the tetrad fields at the boundary in terms of which the boundary conditions are formulated, some discussions were provided to motivate the use of "radiation controlling" boundary conditions, which control the incoming gravitational radiation (in a sense that can be made precise for the linearized equations). This gives rise to a family of constraint-preserving boundary conditions which generalize the typical Sommerfeld-type boundary conditions by allowing a certain coupling in the first-derivatives of the fields. Next, it was shown how to obtain energy estimates for these generalized Sommerfeld conditions on which the (local-in-time) well posedness result for the harmonic Einstein system is based. The talk ended with brief comments regarding open issues related to geometric uniqueness and regarding the construction of higher-order absorbing boundary conditions.

Steven Lau described ongoing work towards the construction (via multidomain, modal, spectral methods) of helically symmetric spacetimes representing binary configurations. While unphysical, such solutions would be of mathematical interest; for example, they would have pathological asymptotic structure. The approach starts with the helical reduction of the harmonic-gauge Einstein equations proposed in [3]. These reduced field equations can be considered either with fluid interiors or internal boundary conditions determined by moving particles. The former scenario is the primary interest, but Lau's talk focused on the particle models since they involve fewer technical details. Lau demonstrated that the reduced Einstein equations can be solved to high accuracy by the described numerical methods. However, the resulting (numerical) solutions violate the harmonic gauge and, therefore, do not correspond to solutions of the Einstein equations. Lau identified incorrect boundary conditions as the culprit responsible for the gauge violation. After the talk, D. Hilditch suggested another scenario that might help in improving the boundary conditions; namely, a "head-on" (no rigid rotation) configuration of neutron stars which is helically symmetric through a balance of outgoing and incoming radiation (helically symmetric configurations with rotation also rely on radiation balance). Discussions with other participants also raised the possibility of applying the described numerical methods to the problem of matching a matter interior solution to the Kerr exterior.

Jörg Frauendiener presented recent work by him and his research group (see [5]) based on a global approach to describe the nonlinear interaction of gravitational waves. The cleanest description is based on a certain conformal invariance of the Einstein equations – a fact which was established by R. Penrose and was used by H. Friedrich to prove several important global results for general relativistic space-times. The so called conformal field equations implement this conformal invariance on the level of partial differential equations. They provide various well-posed initial (boundary) value problems for use in different situations. Frauendiener's talk provided a computational perspective on the nonlinear interaction of plane gravitational waves and also presented preliminary results of a simulation of the behaviour of an initially spherically symmetric black hole under the impact of a gravitational wave burst.

Jacques Smulevici provided a review on the status of the IBVP in GR and geometric uniqueness. In the absence of time-like boundary, the classical initial value problem in GR verifies a geometric uniqueness property. In particular, isometric Cauchy data leads to (maximal globally hyperbolic) developments which are isometric. While there exists several well-posed formulation of the IBVP in GR, no such geometric uniqueness is known. This important issue was put forward by H. Friedrich. It is relevant not only for the local IBVP, but also for more global aspects, due to the possible breakdown of gauge choices. In his talk, Smulevici reviewed the mathematical analysis of the IBVP in GR, with an emphasis on various aspects relevant to the geometric uniqueness problem. Furthermore, some promising preliminary results (with Grigorios Fournodavlos) were given concerning an approach to the IBVP based on the wave equation satisfied by the second fundamental form of a foliation with prescribed mean curvature.

Luisa Buchman talked about implementing higher-order absorbing boundary conditions for numerical relativity simulations of binary black holes. The numerical computation of gravitational radiation emitted from such systems has been pivotal in the detection and characterization of 10 binary black hole astrophysical events since Sept. 14, 2015. However, the drive for more accurate waveforms remains, especially when computing higher-order spherical harmonic modes. The numerical relativity simulations currently used for LIGO / VIRGO detections solve Einsteins field equations on a finite computational domain with an artificial outer boundary. In order to obtain a unique Cauchy evolution, it is necessary to impose boundary conditions which are, ideally, well-posed, constraint-preserving, and completely transparent to the physical problem on the unbounded domain. This last point, however, is still an open issue. In 2006, Buchman and Sarbach [8] developed a hierarchy of absorbing boundary conditions which form a well-posed IBVP, are constraint preserving, and which insure that the spurious gravitational radiation reflected from the outer boundary into the computational domain is minimal. These boundary conditions, named BL, have been implemented in the dual-frame, multi-grid Spectral Einstein Code (SpEC) for a first order generalized harmonic formulation of the Einstein equations. BL have been tested in numerical evolutions of both multipolar wave and binary black hole initial data. Numerical reflection coefficients were calculated from either Regge-Wheeler-Zerilli [41, 47] or Newman-Penrose scalars and compared with theoretical predictions. For multipolar wave initial data and for reflection coefficients obtained from Regge-Wheeler-Zerilli scalars extracted at the boundary during the evolution, the numerical and theoretical predictions match to the level expected for all multipoles and boundary condition orders tested. However, when the numerical reflection coefficients are computed using extracted Newman-Penrose scalars, they match the theoretical predictions for only the lowest order boundary condition. For binary black hole inspirals, the numerical and theoretical reflection coefficients did not match for any of the cases tested. After her talk, Buchman received useful feedback and potential collaborations formed so that this work can move forward and, hopefully, soon be completed.

David Hilditch talked about a parametrized formulation of Einstein's vacuum equations which reduces in some special cases to some of the formulations used in the numerical relativity community. Some recent results concerning boundary conditions which give rise to a well-posed IBVP were given [22].

3.3 AdS spacetimes

Jonathan Luk discussed joint work with G. Holzegel, S. Smulevici and C. Warnick [23] on the global dynamics of the wave equation, Maxwell's equation and the linearized Bianchi equations on a fixed AdS background. Provided dissipative boundary conditions are imposed on the dynamical fields, uniform boundedness of the natural energy as well as both degenerate (near the AdS boundary) and non-degenerate integrated decay estimates can be proven. Remarkably, the non-degenerate estimates lose a derivative. This loss is related to a trapping phenomenon near the AdS boundary, which itself originates from the properties of (approximately) gliding rays near the boundary. Using the Gaussian beam approximation it was proven that non-degenerate energy decay without loss of derivatives does not hold. As a consequence of the non-degenerate integrated decay estimates, pointwise-in-time decay estimates for the energy have been obtained. In an ongoing project, there is hope to establish the global nonlinear stability of AdS for the Einstein vacuum equations under dissipative boundary conditions (without symmetry assumptions). In addition to the nonlinear problem, there are a few interesting related open problems (some of which have been asked during the workshop): i) In [23] hyperbolic equations which are conformally invariant have been studied. What happens for instance for the Klein–Gordon equation with a non-conformal mass? ii) Understand the asymptotic behavior of the linearized

Bianchi equations for the full class of boundary conditions allowed in the nonlinear local existence result of Friedrich. iii) Understand the quasinormal modes associated to this problem.

Oleg Evnin reviewed nonlinear dynamics in AdS spacetimes, and its effective description in terms of resonant Hamiltonian systems. AdS spacetimes have a peculiar property that differences of any two normal mode frequencies for their linearized perturbations are integer in appropriate units. As a result, there is a profusion of resonances, and arbitrarily small nonlinearities may produce arbitrarily large effects, provided that one waits long enough. This situation is closely paralleled by nonlinear Schrödinger equations in harmonic traps, studied in relation to the physics of cold atomic gases and connected to the AdS dynamics via a nonrelativistic limit. At leading order, weakly nonlinear long-time dynamics of this sort is captured by the corresponding resonant Hamiltonian systems. Evnin described the derivation of this resonant approximation and its remarkable structures, including manifestations of turbulent behaviors, as well as symmetry enhancements and explicit analytic solutions. He stated the state-of-the-art of these questions, and then pointed out a few significant outstanding problems, such as analyzing the turbulent blow-up in solutions of the AdS5-scalar-field resonant system with full gravitational backreaction in spherical symmetry (for which there is strong numerical evidence), and better understanding of the classes of explicit analytic solutions emerging in some of the considered resonant systems, as well as finding new explicit solutions. Underway, an interesting discussion with Jonathan Luk has emerged on whether understanding of AdS instabilities can be attained from general collapse arguments in GR, or whether detailed dynamical studies within the resonant approximation are crucial. Evnin supports the latter view.

Andrzej Rostworowski talked on a new perspective on metric perturbations and AdS geons. The key result of Schwarzschild black hole perturbation theory is that at linear level the general perturbation can be given in terms of only two (axial/polar) master scalars satisfying scalar wave equation on the Schwarzschild background [35] with Regge-Wheeler [41] and Zerilli [47] potentials for axial and polar sectors respectively (more precisely, this holds for any multipole $\ell \geq 2$; the monopole $\ell = 0$ and dipole $\ell = 1$ cases need some special treatment). This remarkable result is usually obtained by tedious manipulations o the linearized Einstein equations (cf. [47]). Rostworowski has shown that this classical result can be easily obtained starting with the ansatz that all gauge invariant characteristics of perturbations (for a given $\ell > 1$ multipole) are given in terms of a master scalar and its derivatives, where the master scalar satisfies a scalar wave equation (with a potential) on the background solution. This new perspective can be easily extended beyond the linear approximation [42] and it was used to provide the evidence for the existence of globally regular, asymptotically-AdS, time-periodic solutions of Einstein equations [43]. It can be also easily generalized to include matter, either in the form of some fundamental fields (studied in the AdS/CFT context) or effective perfect fluid approximation (for example in the context of cosmological perturbations [44]).

Arick Shao spoke about correspondence and rigidity results on asymptotically AdS spacetimes. In theoretical physics, it is often conjectured that a correspondence exists between the gravitational dynamics of asymptotically AdS spacetimes and a conformal field theory of their boundaries. In the context of classical relativity, one can attempt to rigorously formulate such a correspondence statement as a unique continuation problem for PDEs: Is an asymptotically AdS solution of the Einstein equations uniquely determined by its data on its conformal boundary? Shao's talk reported on recent progress in this direction (which is mostly joint work with Gustav Holzegel [24]). In particular, the talk highlighted connections between correspondence conjectures in physics, unique continuation theory for wave equations, and the geometry of asymptotically AdS spacetimes. The first part of the presentation provided a precise statement of the correspondence problem. One point of emphasis was on Fefferman-Graham expansions from the conformal boundary, with discussions regarding the extent that these expansions held for generic Einstein-vacuum spacetimes, as well as how they are related to conformal boundary data for the Einstein equations. The next part surveyed some recent unique continuation theorems for wave equations on asymptotically AdS spacetimes, which form a key step toward correspondence results for the full Einstein equations. One novel element of these theorems is that one must put data on a sufficiently large time interval on the conformal boundary in order for unique continuation to hold, even locally. Moreover, this large time requirement can be connected to the behavior of null geodesics near the conformal boundary. The last portion of Shao's presentation focused on work in progress. In particular, this included ongoing work on the main correspondence problem, as well as progress on a related symmetry extension problem – roughly, whether a symmetry of the conformal boundary necessarily extends into a symmetry of the bulk spacetime.

Christoph Kehle presented his recent results publish in [27] on solutions to the massive linear wave equation $\Box\Psi + m\Psi = 0$ on the interior of Reissner-Nordström-AdS black holes. This is motivated by the Strong Cosmic Censorship Conjecture for asymptotically AdS black holes with negative cosmological constant $\Lambda < 0$. The main result shows that linear waves arising from a space-like hypersurface with Dirichlet (reflecting) boundary conditions imposed at infinity remain bounded in the interior and can be extended continuously beyond the Cauchy horizon. This result is surprising because in contrast to black hole backgrounds with non-negative cosmological constant, the decay of Ψ in the exterior region for asymptotically AdS black holes is only logarithmic (compared to the polynomial and exponential decays in the $\Lambda = 0$ and $\Lambda > 0$ cases, respectively). Its proof relies on a new approach, combining physical space estimates with Fourier based estimates, partly based on the scattering theory developed in [28]. The last part of Kehle's talk was devoted to an outlook on ongoing work on Kerr–AdS. In contrast to Reissner-Nordström-AdS, resonances and instabilities associated to the Cauchy horizon are no longer decoupled from the trapping in the exterior. Indeed, for a specific set of black hole parameters (M, a, Λ), smooth and compactly supported data lead to an L^∞ blow-up at the Cauchy horizon, supporting the validity of the C^0-formulation of Strong Cosmic Censorship.

3.4 Progress in related subjects

Robert Oeckl talked about quantization on time-like hypersurfaces in curved spacetime. When investigating quantum or semiclassical phenomena in a general relativistic context one usually resorts to quantum field theory in curved spacetime. However, traditional quantization methods rely on foliating spacetime into space-like hypersurfaces or on fixing asymptotic boundary conditions in time. This is a serious limitation if for the problem of interest no suitable Cauchy hypersurfaces exist and/or if boundary conditions are naturally given on time-like hypersurfaces, either at finite locations or asymptotically. In his talk, Oeckl outlined methods of quantization adapted to such situations and the conceptual insights on which they are based. The underlying framework is General Boundary Quantum Field Theory (GBQFT) [38], and one of its key ingredient is the insight that transition amplitudes (associated to time intervals in spacetime) are merely special cases of amplitudes (associated to general spacetime regions). This leads to a corresponding generalization of the S-matrix to a potentially much larger class of asymptotic as well as finite-region scattering amplitudes. For a comprehensive account of how this fits into the foundations of physics, see [39].

Stephen Green presented some recent result (together with S. Hollands and P. Zimmerman [19]) on the orthogonality of Kerr quasinormal modes. For linear perturbations of black hole spacetimes, the ringdown is described by quasinormal modes. Quasinormal modes are resonance states, defined by ingoing and outgoing radiation conditions at the horizon and infinity. Their wavefunctions do not lie in a Hilbert space, and they do not in general form a complete basis. In particular, there is a priori no obvious inner product under which they are orthonormal. In contrast to normal modes, this limits efforts to carry out higher order perturbation theory in terms of quasinormal modes. In his work, Green showed that for type D spacetimes with a $t - \phi$ reflection symmetry, gravitational quasinormal modes are in fact orthogonal with respect to an appropriate symmetric bilinear form $\langle \cdot, \cdot \rangle$ which is defined in terms of the symplectic form for the Teukolsky equation on a Cauchy surface. This bilinear form is complex-linear in both entries, it is independent of the precise choice of the Cauchy surface, and the time-evolution operator is symmetric with respect to $\langle \cdot, \cdot \rangle$. It follows that quasinormal modes are orthogonal with respect to $\langle \cdot, \cdot \rangle$ and have finite norm. Green also related the bilinear form on Weyl scalars to bilinear forms on the spaces of metric perturbations and Hertz potentials. The formula for a quasinormal mode excitation coefficient emerges naturally as the projection of initial data onto the quasinormal mode, however this projector also extends to data consisting of quasinormal modes. By projecting with the bilinear form, the goal is to develop a framework for higher order black hole perturbation theory in terms of quasinormal modes.

Helvi Witek gave an overview of the recent progress on simulating binary collisions and computing the expected gravitational radiation in theories that extend GR (such theories are partially motivated by the need of incorporating high-energy scale effects). Modeling the expected gravitational radiation in these extended theories enables one to search for – or place novel observational bounds on – deviations from the standard GR model. Recent progress on simulating binary collisions in these situations as well as renewed mathematical challenges such as well-posedness of the underlying initial value formulation were addressed.

Eloisa Bentivegna talked about the problem of exploring large computational models, which is generally a complex task, requiring the efficient inspection of high-dimensional parameter spaces. This is exemplified by the task of identifying which combinations of binary-black-hole spin and mass ratio values lead to specific gravitational wave profiles, or which cosmological matter distribution produces a certain weak-lensing signature. The challenges involved are similar to those encountered in system identification and nonlinear control theory, where the behaviour of a system as a function of its configuration variables is reconstructed based on the direct, optimal sampling of the space of all possible system trajectories. In her talk, Bentivegna described the application of an exploration technique, proposed to probe rare solutions of certain ODEs, to the exploration of null geodesics in simple spacetimes, and also discussed the complementarity of this method to other model exploration techniques which are widely used in GR, such as reduced-order modeling.

4 Scientific Progress Made and outcome of the Meeting

Besides the talks, the workshop provided the opportunity for plenty of discussion sessions. There was one discussion session on the artificial boundary problem. Friedrich immediately pointed out key difficulties in constructing artificial boundary conditions (ABCs) for GR, for example, (i) the impossibility in the nonlinear regime of unambiguously separating the solution into incoming and outgoing waves and (ii) gauge issues in choosing a finite boundary. Approaches based on linearization (either about flat or black hole spacetimes) are one possibility, and one such approach had been presented in Buchman's talk. Sarbach then asked about corresponding ABCs for other wave equations, and Lau pointed out that most work thus far has been for linear problems, and therefore not immediately extendable to GR (without some type of linearization). Nonetheless, while numerical strategies for rapid implementation of ABC are often local in nature (at least in time), ABC design for the wave and Maxwell equations relies on the nonlocal and history-dependent conditions specifying exact domain reduction. Lau stressed that such nonlocality should also be present for any exact domain reduction in GR. He then asked about the status of "hyperboloidal layers", which is conceptually the cleanest approach to ABC in GR. Frauendiener commented on the technical difficulties inherent in the approach, in particular on the fact that the key troubles occur (at future null infinity) precisely where gravitational wave signals need to be read-off. Frauendiener discussed the possibility of imposing boundary conditions "beyond null infinity", and indeed had shown in his talk that this is possible, although perhaps not yet in a suitably general setting. Informal discussions during the coffee breaks also focused on possible routes toward circumventing technical obstacles associated with hyperboloidal layers.

Various discussions also took place surrounding IBVPs in the context of AdS spacetimes. One topic of interest concerned the spherically symmetric $\Lambda < 0$ Einstein-Klein-Gordon equation with reflecting boundary conditions, and the somewhat complicated picture that appears to emerge concerning the fate of perturbations of AdS: instability; islands of stability; (quasi-)periodic solutions. It appears likely that some progress will be made in the near future towards the construction of time-periodic solutions. The topic of geometric boundary conditions for the vacuum $\Lambda < 0$ Einstein equations was also discussed. For some of the boundary conditions considered by Friedrich in his seminal well-posedness paper [14] there is a natural geometric formulation leading to a geometric uniqueness result, while for others such a geometric formulation is currently unavailable. This problem is connected to the related one on a finite domain. Some interesting ideas were discussed in both contexts and there is hope progress will be made in the near future towards an understanding of the geometric uniqueness problem.

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