Muchnik degrees and cardinal characteristics

(short version)

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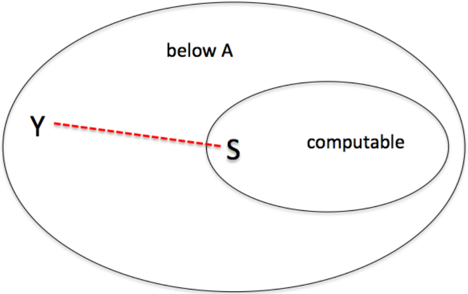
- For $Z \in 2^{\mathbb{N}}$ let $\overline{\rho}(Z) = \limsup_n \frac{|Z \cap [0,n]|}{n}$ (upper density).
- For $X, Y \in 2^{\mathbb{N}}$ let $d(X,Y) = \overline{\rho}(X \triangle Y)$ (Besicovich pseudo distance).

Definition (Andrews et al., 2013, rephrased)

Given an oracle set A let

$$c(A) := \mathbf{d}_H(\{Y : \ Y \leq_{\mathrm{T}} A\}, \text{computable})$$

where $\mathbf{d}_H$ is Hausdorff distance $\sup_{Y \leq_T A} \inf_{S \text{ comp.}} d(Y,S))$.



Recall: $c(A) = \mathbf{d}_H(\{Y : Y \leq_T A\}, \text{computable})$.
$c(A) < 1/2 \Leftrightarrow A$ computable. $c(A) = 1 \Leftarrow A$ hyperimmune.
$c(A) = 1/2$ e.g. certain random sets.

Their Γ-question asked whether $c(A) > 1/2$ implies $c(A) = 1$.

Theorem (Monin (2016), Logic in Computer Science 2018)

$c(A)$ is either 0, or $1/2$, or 1.
Also $c(A) = 1 \Leftrightarrow \exists f \leq_T A$
$\forall g$ computable, bounded by $2^{(2^n)} \exists^\infty n\ f(n) = g(n)]$

- Brendle and N. (2014) in Logic Blog 2015 entry introduced parameterized cardinal invariants $\mathfrak{b}(p), \mathfrak{d}(p)$ for $p \in [0, 1]$, and discussed classes $\mathfrak{b}(\neq_h^*), \mathfrak{d}(\neq_h^*)$
- Monin and N. (Logic in Computer Science conference, 2015) connected their recursion theoretic analogs.
- Monin and N. journal paper (submitted) provides dual of the recursion theoretic result and does the analogous ZFC equalities for cardinal characteristics.

Cardinal characteristics and their analogs

Rupprecht in his 2010 thesis studied computability theoretic analogs of cardinal characteristics. We have a binary relation $R \subseteq \mathcal{X} \times \mathcal{Y}$ between sets, or functions. Recall

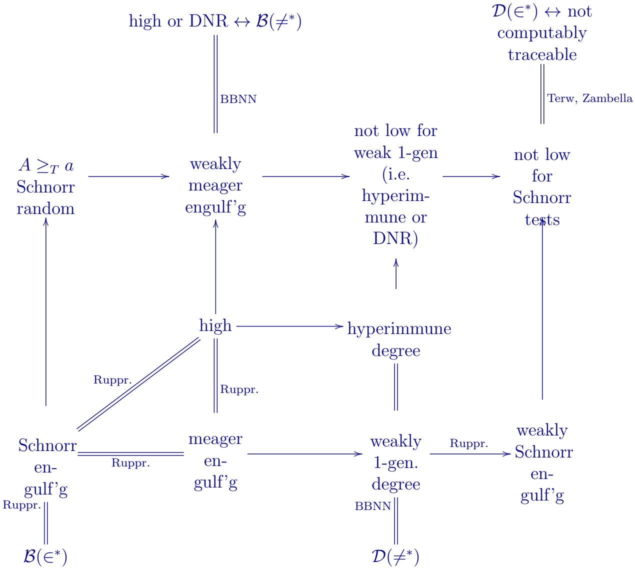
$$\begin{aligned} \mathfrak{b}(R) &= \min\{|F| : F \subseteq X \wedge \forall y \in Y \exists x \in F \, \neg xRy\} \\ \mathfrak{d}(R) &= \min\{|G| : G \subseteq Y \wedge \forall x \in X \, \exists y \in G \, xRy\}. \end{aligned}$$

Variable x ranges over X, and y ranges over Y. One defines the analogous highness properties of Turing oracles

$$\begin{aligned} \mathcal{B}(R) &= \{A : \exists y \leq_{\mathrm{T}} A \, \forall x \text{ computable } [xRy]\} \\ \mathcal{D}(R) &= \{A : \exists x \leq_{\mathrm{T}} A \, \forall y \text{ computable } [\neg xRy]\}. \end{aligned}$$

Note we are negating the set theoretic definitions. Reason: to "increase" a cardinal of the form $\min\{|F| : \phi(F)\}$, we need to introduce via forcing objects y so that $\phi(F)$ no longer holds in an extension model. This forcing corresponding to the construction of a powerful oracle computing a witness for $\neg\phi$.

Analog of Cichoń's diagram (Rupprecht '10, BBNN, '14)



Relations $\neq^*_h$ and $\bowtie_p$

Let $h\colon \omega \to \omega - \{0,1\}$. For $x \in {}^\omega\omega$ and $y \in \prod_n \{0,\ldots,h(n)-1\} \subseteq {}^\omega\omega$, let

$$x \neq^*_h y \Leftrightarrow \text{ a.e. } n\,[x(n) \neq y(n)].$$

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Let $\underline{\rho}(z) = \liminf_n |z \cap n|/n$ for a bit sequence z.
Let $0 \le p < 1$. For $x, y \in {}^\omega 2$ let

$$x \bowtie_p y \Leftrightarrow \underline{\rho}(x \leftrightarrow y) > p,$$

where $x \leftrightarrow y$ is the set of n such that $x(n) = y(n)$.

Equalities in computability theory

$\mathcal{D}(\neq_h^*)$: A computes a function y such that for each computable function $x < h$, one has $\exists^\infty n\, x(n) = y(n)$.
$\mathcal{D}(\bowtie_p)$: A computes a bit sequence y such that for each computable set x, one has $\underline{\rho}(x \leftrightarrow y) \leq p$.
$\mathcal{B}(\bowtie_p)$: A computes a bit sequence x such that for each computable set y, one has $\underline{\rho}(x \leftrightarrow y) > p$.
The following uses the techniques of Monin (2016) and dualises them as well.

Theorem (Monin and N., 2017)

Fix any $p \in (0, 1/2)$. We have

$$\mathcal{D}(\bowtie_p) = \mathcal{D}(\neq^*, (2^{(2^n)})) \text{ and } \mathcal{B}(\bowtie_p) = \mathcal{B}(\neq^*, (2^{(2^n)})).$$

The proof is via several intermediate classes.
Function values are viewed as encoding strings; this is where the double exponential comes from.

View the highness properties as mass problems

- Instead of classes of Turing oracles we use so-called "mass problems" (i.e. subsets of ω^ω).
- They are compared via Muchnik (or weak) reducibility: $\mathcal{C} \leq_W \mathcal{D}$ if $\forall Y \in \mathcal{D} \exists X \in \mathcal{C}\ X \leq_T Y$.

Re-define

$\mathcal{B}(\bowtie_p) = \{X \in 2^{\mathbb{N}} \colon \forall Y \text{ computable}, \underline{\rho}(X \leftrightarrow Y) > p\}$.

$\mathcal{B}(\neq^*, h) = \{f < h \colon \forall g \text{ computable } \text{ a.e. } n\, [g(n) \neq f(n)]\}$.

Let $\leq_S$ denote uniform reducibility, where the oracle TM is fixed. For the case of $\mathcal{B}$ we have uniform reductions.

Theorem (strengthens half of previous theorem)

$\mathcal{B}(\bowtie_p) \equiv_S \mathcal{B}(\neq^*, 2^{(2^n)})$ for each $p \in (0, 1/2)$.

ZFC equalities

$\mathfrak{d}(\neq^*_h)$ is the least size of a set G of h-bounded functions so that for each function x there is a function y in G such that $\text{a.e.} n\, [x(n) \neq y(n)]$.

$\mathfrak{d}(\bowtie_p)$ is the least size of a set G of bit sequences so that for each bit sequence x there is a bit sequence y in G so that $\underline{\rho}(x \leftrightarrow y) > p$.

Theorem (Monin and N., 2017)

Fix any $p \in (0, 1/2)$. We have

$$\mathfrak{d}(\bowtie_p) = \mathfrak{d}(\neq^*, (2^{(2^n)}) \text{ and } \mathfrak{b}(\bowtie_p) = \mathfrak{b}(\neq^*, (2^{(2^n)}).$$

Question

Is it consistent with ZFC to have $\mathfrak{d}(0) < \mathfrak{d}(1/4)$?
To have $\mathfrak{b}(0) > \mathfrak{b}(1/4)$?

Separations of hierarchies

An order function is a function $G\colon \mathbb{N} \to \mathbb{N}$ that is recursive, nondecreasing unbounded.

Theorem (Joe Miller, Khan; Khan and N.)

Let $F, G \in {}^{\omega}\omega$ be order functions such that $G >> F$.
Then $\mathcal{B}(\neq^*, G) \supset \mathcal{B}(\neq^*, F)$ (proper containment).

Khan and Miller used forcing with bushy trees to separate classes of bounded DNR functions. Khan and N. showed these classes correspond to classes $\mathcal{B}(\neq^*, .)$ with similar bounds.
Analog in set theory: Kamo - Osuga 2011.

Theorem (Joe Miller, Monin, N.)

Let $F, G \in {}^{\omega}\omega$ be order functions such that $G >> F$.
Then $\mathcal{D}(\neq^*, G) \subset \mathcal{D}(\neq^*, F)$.

E.g., if $F(n) = n$ we can let $G(n) = \exp\exp\exp(n^2)$.
Analog in set theory not known at present.

References

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