Hodge elliptic genera in geometry and in CFT

Geometric and Categorical Aspects of CFTs,

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Plan:

1. Refining the Euler characteristic
2. More algebra and mathematical physics
3. Interpretation in CFT

[W17] *Hodge-elliptic genera and how they govern K3 theories*; arXiv:1705.09904 [hep-th]

[Taormina/W15] *A twist in the M_{24} moonshine story*, Confluentes Mathematici 7, 1 (2015), 83-113; arXiv:1303.3221 [hep-th]

[Taormina/W13] *Symmetry-surfing the moduli space of Kummer K3s*, Proceedings of Symposia in Pure Mathematics 90 (2015), 129-153; arXiv:1303.2931 [hep-th]

[Taormina/W11] *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, JHEP **1308**:152 (2013); arXiv:1107.3834 [hep-th]

1. Refining the Euler characteristic: Complex elliptic genus

Euler characteristic:

$$\chi(M) := \sum_{j,k} (-1)^{j+k} h^{k,j}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*) = \chi(\bigoplus_k (-1)^k \Lambda^k T^*)$$

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for any bundle $E \to M$, $\Lambda_x E := \bigoplus_{k=0}^{\infty} x^k \Lambda^k E$, $S_x E := \bigoplus_{k=0}^{\infty} x^k S^k E$

Hirzebruch χ_y-genus:

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Definition [Hirzebruch88, Witten88]

With $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$ for $\tau, z\in\mathbb{C}$, $\mathrm{Im}(\tau)>0$,

$$\mathbb{E}_{q,-y} := y^{-\frac{D}{2}}\Lambda_{-y}T^*\otimes\bigotimes_{n=1}^{\infty}\left[\Lambda_{-yq^n}T^*\otimes\Lambda_{-y^{-1}q^n}T\otimes S_{q^n}T^*\otimes S_{q^n}T\right],$$

$$\mathcal{E}(M;\tau,z) := \chi(\mathbb{E}_{q,-y}) = \int_M \mathrm{Td}(M)\,\mathrm{ch}(\mathbb{E}_{q,-y})$$

is the COMPLEX ELLIPTIC GENUS of M.

1. Properties of the complex elliptic genus

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  a weak Jacobi form (weight 0, index $\frac{D}{2}$) with respect to $SL_2(\mathbb{Z})$
- it's a genus with values in the ring of weak Jacobi forms of weight 0

1. Refining $\chi(M)$: The (geometric) Hodge elliptic genus

Definition

$\mathbb{E}_{q,-y}$ as before, $\mathbb{E}_{q,-y} = y^{-\frac{D}{2}} \bigoplus_{\ell,m} q^\ell (-y)^m \mathcal{T}_{\ell,m}$,

COMPLEX ELLIPTIC GENUS of M:

$$\mathcal{E}(M;\tau,z) = \quad y^{-\frac{D}{2}} \sum_{\ell,m} q^\ell (-y)^m \sum_j (-1)^j \dim H^j(M, \mathcal{T}_{\ell,m}).$$

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Theorem [Kachru/Tripathy16]

If M is a complex torus or a K3 surface, then $\mathcal{E}^{\mathrm{HEG}}(M;\tau,z,\nu)$ is an invariant (that is, independent of the complex structure).

2. Attaching vertex operator algebras to M

The basic building block: $bc - \beta\gamma$ system E

$\mathfrak{a}$: (complex) Heisenberg algebra with basis $(\beta_n,\ \gamma_m,\ 1)_{n,m\in\mathbb{Z}}$,

$\forall n, m \in \mathbb{Z}$: $[\beta_n, \gamma_m] = \delta_{n+m,0} \cdot 1$, and all other $[x_n, y_m] = 0$

$\mathfrak{a}_-$: sub Lie algebra with basis $(\beta_n,\ \gamma_m,\ 1)_{n\leq 0, m<0}$

$\underline{\mathbb{C}} := \mathrm{span}_{\mathbb{C}}(\Omega)$, $\quad \forall n \leq 0\colon \beta_n.\Omega = 0$, $\forall m < 0\colon \gamma_m.\Omega = 0$, $1.\Omega = \Omega$

$$F := \mathrm{ind}_{\mathfrak{a}_-}^{\mathfrak{a}}(\underline{\mathbb{C}}) \cong \mathbb{C}[\beta_1,\ \beta_2,\ \beta_3,\ \ldots, \gamma_0,\ \gamma_1,\ \gamma_2,\ \ldots]$$

F carries the structure of a vertex operator algebra (VOA),
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introducing free fermions along the same lines, get a Fock space $E \supset F$,
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For $U \subset M$: holomorphic coordinate chart with $\mathbb{E}_{q,-y|U} \cong U \times \mathbb{E}$,
$\mathbb{E}$ a super-module of the super-VOA $E^{\otimes D}$.

[Dong/Liu/Ma02], using the $\mathrm{SU}(D)$-holonomy of M, obtain
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BUT: In TQFT, we need to include the zero modes $\gamma_0^{(j)}$.

2. The chiral de Rham complex

Definition **[Malikov/Schechtman/Vaintrob99]**
CHIRAL DE RHAM COMPLEX Ω_M^{ch}: sheaf of super-VOAs over M,
for any holomorphic coordinate chart $U \subset M$: $\Omega_M^{\mathrm{ch}}(U) := E^{\otimes D}$.

Theorem **[Malikov/Schechtman/Vaintrob99; Borisov/Libgober00]**
$H^*(M, \Omega_M^{\mathrm{ch}})$ (sheaf cohomology) is a topological $N = 2$ superconformal VOA.
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Consequence:

$$\mathcal{E}(M;\tau,z) = y^{-\frac{D}{2}} \sum_j (-1)^j \underbrace{\mathrm{tr}_{H^j(M,\Omega_M^{\mathrm{ch}})}\left((-y)^{J_0} q^{L_0^{\mathrm{top}}}\right)}_{\neq\ \text{gr-dim}\left(H^j(M,\mathbb{E}_{q,-y})\right),\ \text{in general}},$$

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Results **[W17]** on $\mathcal{E}^{\mathrm{HEG,ch}}(M;\tau,z,\nu)$ (using **[Creutzig/Höhn14, Song16]**):
If M is a complex torus, then $\mathcal{E}^{\mathrm{HEG,ch}}(M;\tau,z,\nu)$ agrees with $\mathcal{E}^{\mathrm{HEG}}(M;\tau,z,\nu)$;
if M is a K3 surface, then it is an invariant, different from $\mathcal{E}^{\mathrm{HEG}}(M;\tau,z,\nu)$.

3. Interpretation in (2d unitary, Euclidean) SCFT

Fact:

$\mathcal{C}$: a superconformal field theory (SCFT) at central charge $c = 3D$, $D \in \mathbb{N}$
(assuming $N = (2,2)$ worldsheet SUSY and spacetime SUSY)

$\Longrightarrow$ commuting J_0, $\underbrace{L_0^{\mathrm{top}}}_{L_0 - \frac{1}{2}J_0}$, $\widetilde{J}_0$, $\underbrace{\widetilde{L}_0^{\mathrm{top}}}_{\widetilde{L}_0 - \frac{1}{2}\widetilde{J}_0}$ act on the space of states,

as well as $\mathcal{A}$, an extended $N = 2$ SCA with $c = 3D$, $J_0, L_0 \in \mathcal{A}$.

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Then $\mathbb{H} := \ker(\widetilde{L}_0^{\mathrm{top}})$ is an sVOA, and

$$\mathcal{E}_{\mathrm{CFT}}(\mathcal{C};\tau,z) := \mathrm{tr}_{\mathbb{H}}\left((-1)^{J_0-\widetilde{J}_0} y^{J_0-c/6} q^{L_0^{\mathrm{top}}}\right) \quad \in y^{-D/2}\cdot\mathbb{Z}[[q,y^{\pm 1}]]$$

is a weak Jacobi form of weight 0 and index $\frac{D}{2}$, the CFT ELLIPTIC GENUS.

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Expectation:

Such an SCFT $\mathcal{C}$ exists "for every M" as above, $\mathcal{E}_{\mathrm{CFT}}(\mathcal{C}; \tau, z) = \mathcal{E}(M; \tau, z)$.
This **expectation holds true** if M is a complex torus or a K3 surface.

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Definition [Kachru/Tripathy16]
CONFORMAL FIELD THEORETIC HODGE ELLIPTIC GENUS:

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3. Interpretation in CFT

Results

- **[Kapustin05]**: For theories $\mathcal{C}$ associated to M, $\mathbb{H} \xrightarrow{\text{large volume}} H^*(M, \Omega_M^{\text{ch}})$.

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- **[Kapustin05]**: For theories $\mathcal{C}$ associated to M, $\mathbb{H} \xrightarrow{\text{large volume}} H^*(M, \Omega_M^{\mathrm{ch}})$.
- For K3 theories $\mathcal{C}$ ($c = 6$):

  Let $\mathbb{H}_0 :=$ the GENERIC SPACE OF STATES, i.e. maximal such that at every point of the moduli space, $\mathbb{H}_0 \hookrightarrow \mathbb{H}$ as a representation of $\langle \mathcal{A}, \widetilde{J}_0 \rangle$,

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Results

- **[Kapustin05]**: For theories $\mathcal{C}$ associated to M, $\mathbb{H} \xrightarrow{\text{large volume}} H^*(M, \Omega^{ch}_M)$.
- For K3 theories $\mathcal{C}$ ($c = 6$):

  Let $\mathbb{H}_0 :=$ the GENERIC SPACE OF STATES, i.e. maximal such that at every point of the moduli space, $\mathbb{H}_0 \hookrightarrow \mathbb{H}$ as a representation of $\langle \mathcal{A}, \widetilde{J}_0 \rangle$,

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Open:

- Is any VOA structure of $\mathbb{H}_0$ compatible with the M_{24}-action?
- Is M_{24} generated by symmetry surfing,

  as suggested in **[Taormina/W11$+\cdots$]**?

Thank you
for your attention!