General-order observation-driven models

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Outline

Introduction

Statistical inference

Back to consistency

Conclusion

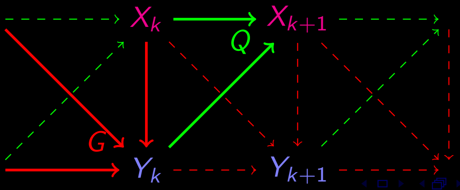
Partially observed Markov models

Definition (Partially observed Markov model (POMM))

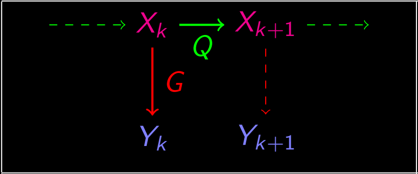
A partially observed Markov model with latent space $(\mathsf{X}, \mathcal{X})$ and observation space $(\mathsf{Y}, \mathcal{Y})$ is a pairwise homogeneous Markov chain $((Y_n, X_n), \mathcal{F}_n)_{n \geqslant 0}$ with kernel $\mathbf{K}^\theta$, $\theta \in \Theta$, generally described as

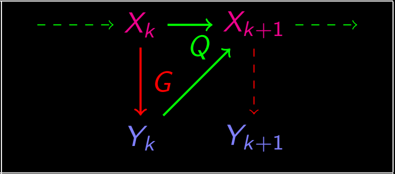
$$\begin{aligned} Y_k | \mathcal{F}_{k-1}, X_k &\sim G^\theta(X_{k-1}, Y_{k-1}, X_k; \cdot) \, , \\ X_{k+1} | \mathcal{F}_k &\sim Q^\theta(X_k, Y_k; \cdot) \, , \end{aligned} \tag{1}$$

and such that only the $\{Y_k\}'$s are observed.



Two important examples





Hidden Markov models (HMM)	Obs.-Driv. models (ODM)

- In both cases, $\{X_k\}$ is a Markov chain.
- An ODM moreover requires that

$$Q^\theta(X_k, Y_k; \cdot) = \delta_{\psi^\theta_{Y_k}(X_k)} ,$$

where δ_x denotes the Dirac mass at point x and, for all $y \in \mathsf{Y}$,

$$\begin{aligned} \psi^\theta : \mathsf{Y} \times \mathsf{X} &\to \mathsf{X} \\ (y, x) &\mapsto \psi^\theta_y(x) . \end{aligned}$$

General-order ODM

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Definition (ODM(p, q))

Let $p, q \geqslant 1$ and, for all $\theta \in \Theta$,

$$\begin{aligned} \psi^\theta \ : \ \mathsf{Y}^p \times \mathsf{X}^q &\to \mathsf{X} \\ (\mathbf{y}, \mathbf{x}) &\mapsto \psi^\theta_{\mathbf{y}}(\mathbf{x}) \ . \end{aligned}$$

An ODM of order (p, q) with link function ψ^θ and observation kernel G^θ satisfies, for all $k \in \mathbb{N}$,

$$\begin{aligned} X_{k+1} &= \psi^\theta_{Y_{(-p+1+k):k}}\left(X_{(-q+1+k):k}\right) \ , \\ Y_{k+1} | \mathcal{F}_k, X_{k+1} &\sim G^\theta\left(X_{k+1}; \cdot\right) \ . \end{aligned} \tag{2}$$

where $\mathcal{F}_k = \sigma(Y_{-p+1}, \ldots, Y_k, X_{-q+1}, \ldots, X_k)$.

Linear ODM

Definition (Linear ODM (LODM))

A linear ODM (LODM) is an ODM

- with parameters $\theta = (\vartheta, \varphi)$ with $\vartheta = (\omega, a_{1:p}, b_{1:q}) \in \mathbb{R}^{1+p+q}$,
- with $\mathsf{X} \subseteq \mathbb{R}$ and link functions of the form

$$\psi^{\theta} : \mathsf{Y}^p \times \mathsf{X}^q \to \mathsf{X}$$

$$(y_{1:p}, x_{1:q}) \mapsto \psi^{\theta}_{\mathbf{y}}(\mathbf{x}) = \omega + \sum_{k=1}^{p} a_k \upsilon(y_k) + \sum_{k=1}^{q} b_k x_k \ , \quad (3)$$

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where $\upsilon : \mathsf{Y} \to \mathbb{R}$.

If $\mathsf{X} = \mathbb{R}_+$, set $(\omega, a_{1:p}, b_{1:q}) \in \mathbb{R}_+^{1+p+q}$ and $\upsilon : \mathsf{Y} \to \mathbb{R}_+$.

Examples of LODM

- GARCH(p, q), Bollerslev [1986]:

$$\mathsf{X} = \mathbb{R}_+ \,, \;\; G^\theta(x; \cdot) = \mathcal{N}(0, x) \quad \text{and} \quad \upsilon(y) = y^2 \,.$$

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Note that, transforming the observations, we can take $\upsilon(y) = y$, but it modifies the definition of G^θ.

Domination

- In contrast to HMMs, ODMs are not fully dominated: $\mathbf{K}^\theta(x, \cdot)$ is not dominated by $(\mu \otimes \nu)$ for σ-finite measures μ and ν on X and Y, resp.
- We always assume that the ODM is partially dominated: there is a σ-finite measure ν on Y such that, for all $\theta \in \Theta$ and $x \in \mathsf{X}$, $G^\theta(x, \cdot)$ is dominated by ν, and, moreover, the density $g^\theta(x; \cdot) = \mathrm{d}G^\theta(x, \cdot)/\mathrm{d}\nu$ satisfies, for all $y \in \mathsf{Y}$,

$$g^\theta(x; y) > 0 ,$$

- To avoid a trivial case, ν is supposed to be non-degenerate, that is, its support contains at least two points.
- For LODMs, we assume the push measure $\nu \circ \upsilon^{-1}$ to be non-degenerate.

Embedding in an ODM(1,1)

Define $\mathsf{Z} = \mathsf{Y}^{p-1} \times \mathsf{X}^q$ and for all $k \in \mathbb{N}$,

$$Z_k = \left(Y_{(k-p+1):(k-1)}, X_{(k-q+1):k}\right) \in \mathsf{Z} \ .$$

- Then $(Y_k, Z_k)_{k \geqslant 0}$ is an ODM(1,1) with link function

$$\begin{aligned} \Psi^\theta &: \mathsf{Y} \times \mathsf{Z} \to \mathsf{Z} \\ &(y, z) \mapsto \Psi^\theta_y(z) = \psi^\theta_{(\mathbf{y}, y)}(\mathbf{x}) \quad \text{where} \quad z = (\mathbf{y}, \mathbf{x}) \ . \end{aligned}$$

- Given an initial distribution η on Z, we denote by $\mathbb{P}^\theta_\eta$ the distribution of $\{X_k, k > -q, \ Y_\ell, \ell > -p\}$ when

$$\left(Y_{(-p+1):-1}, X_{(-q+1):0}\right) \sim \eta \ .$$

Likelihood

- For any $m \in \mathbb{N}$ and $y_{0:m-1} \in \mathsf{Y}^m$, define

$$\Psi^\theta\langle y_{0:m-1}\rangle = \Psi^\theta_{y_{m-1}} \circ \cdots \circ \Psi^\theta_{y_0} \qquad : \mathsf{Z} \to \mathsf{Z}\,, \qquad (4)$$

$$\psi^\theta\langle y_{0:m-1}\rangle = \Pi_{p+q-1} \circ \Psi^\theta\langle y_{0:m-1}\rangle \qquad : \mathsf{Z} \to \mathsf{X}\,, \qquad (5)$$

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- Then, for any arbitrary $z^{(\mathrm{i})} \in \mathsf{Z}$ and observations $Y_{0:n}$, the (conditional) log-likelihood (given $Z_0 = z^{(\mathrm{i})}$) reads

$$\mathsf{L}^\theta_{z^{(\mathrm{i})},n} := \sum_{k=0}^{n} \ln g^\theta\left(\psi^\theta\langle Y_{0:(k-1)}\rangle(z^{(\mathrm{i})});\, Y_k\right)\,. \tag{6}$$

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- Hence the log-likelihood, as well as its derivatives, can easily be computed using $O(n)$ operations.

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- A similar coupling argument yields β mixing properties for (Y_k), see Doukhan and Neumann [2017].

Notation

Recall that, for all $\theta \in \Theta$ and initial distribution η on $(\mathsf{Z}, \mathcal{Z})$, $\mathbb{P}_\eta^\theta$ denotes the distribution of $\{Y_k, X_\ell : k > -p, \ell > -q\}$ for $Z_0 \sim \eta$.

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If the model is ergodic, i.e. there exists a unique probability measure η such that $\mathbb{P}_\eta^\theta$ is shift-invariant, we denote

- the stationary distribution extended on $\left((\mathsf{Y} \times \mathsf{X})^\mathbb{Z}, (\mathcal{X} \otimes \mathcal{Y})^{\otimes \mathbb{Z}}\right)$ by $\mathbb{P}^\theta$,
- the marginalization of $\mathbb{P}^\theta$ on $\left(\mathsf{Y}^\mathbb{Z}, \mathcal{Y}^{\otimes \mathbb{Z}}\right)$ by $\tilde{\mathbb{P}}^\theta$.

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Remark : when establishing ergodicity through a drift condition, we obtain some minimal finite moment condition, referred to as (M) in the following, for X_0 (and Y_0) under the stationary distribution.

Statistical inference

- The Maximum Likelihood Estimator (MLE) $\widehat{\theta}_{z^{(i)},n}$ is defined as

$$\widehat{\theta}_{z^{(i)},n} \in \operatorname{argmax}\left\{\mathsf{L}^{\theta}_{z^{(i)},n} \;:\; \theta \in \Theta\right\} \tag{7}$$

for some arbitrary initial point $z^{(i)} \in \mathsf{Z}$.

- In well-specified models, a standard consistency result consists in showing that

$$\lim_{n\to\infty} \widehat{\theta}_{\eta,n} = \theta_\star\,, \quad \mathbb{P}^{\theta_\star}\text{-a.s.} \tag{8}$$

- and asymptotic normality consists in showing that

$$\sqrt{n}(\widehat{\theta}_{z^{(i)},n} - \theta_\star) \overset{\mathbb{P}^{\theta_\star}}{\Longrightarrow} \mathcal{N}(0, \mathcal{J}^{-1}(\theta_\star)) \tag{9}$$

where $\mathcal{J}(\theta_\star)$ is a nonsingular $d \times d$-matrix.

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This is done by approximating $Y_{0:(k-1)}$ by $Y_{-\infty:(k-1)}$, defined, in the case $k = 1$ by the backward limit

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Let $(\mathsf{X}, \delta_{\mathsf{X}})$ and $(\mathsf{Z}, \delta_{\mathsf{Z}})$ be complete metric spaces in such a way that $\Pi_{p+q+1} : \mathsf{Z} \to \mathsf{X}$ is 1-Lipschitz.

By (4) and (5), $\delta_{\mathsf{X}}\left(\psi^{\theta}\langle Y_{i:0}\rangle(z^{(\mathrm{i})}), \psi^{\theta}\langle Y_{i+1:0}\rangle(z^{(\mathrm{i})})\right)$ is bounded from above by

$$\left(\sup_{y \in \mathsf{Y}^{i}, z, z'} \frac{\delta_{\mathsf{Z}}\left(\Psi^{\theta}\langle y\rangle(z), \Psi^{\theta}\langle y\rangle(z')\right)}{\delta_{\mathsf{Z}}\left(z, z'\right)}\right) \delta_{\mathsf{Z}}\left(\Psi^{\theta}\langle Y_{i+1}\rangle(z^{(\mathrm{i})}), z^{(\mathrm{i})}\right) .$$

where we used a uniform Lipschitz constant of the i- iterate of the link function.

Uniform Lipschitz condition

By (4) and (5), $\delta_{\mathsf{X}}\left(\psi^\theta\langle Y_{i:0}\rangle(z^{(\mathrm{i})}), \psi^\theta\langle Y_{i+1:0}\rangle(z^{(\mathrm{i})})\right)$ is bounded from above by

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Define for all $i \in \mathbb{N}^*$,

$$\mathrm{Lip}_i^\theta = \sup\left\{\frac{\delta_{\mathsf{Z}}\left(\psi^\theta\langle y\rangle(z), \psi^\theta\langle y\rangle(z')\right)}{\delta_{\mathsf{Z}}\left(z, z'\right)} \;:\; y \in \mathsf{Y}^i, z, z' \in \mathsf{Z}\right\} .$$

We use the following condition:

(A-1) For all $\theta \in \Theta$, we have $\mathrm{Lip}_0^\theta < \infty$ and $\mathrm{Lip}_n^\theta \to 0$ as $n \to \infty$,

Asymptotic behavior of the likelihood (cont.)

Under (A-1)+ (M), for all $\theta \in \Theta$, there exists a measurable function $\psi^{\theta}\langle\cdot\rangle : \mathsf{Y}^{\mathbb{Z}_-} \to \mathsf{X}$ such that for all $\theta, \theta_\star \in \Theta$,

$$X_1 = \psi^{\theta}\langle Y_{-\infty:0}\rangle \qquad \mathbb{P}^{\theta}\text{-a.s.}$$

$$\psi^{\theta}\langle Y_{-\infty:0}\rangle = \lim_{n\to\infty} \psi^{\theta}\langle Y_{-n:0}\rangle(z^{(i)}) \qquad \tilde{\mathbb{P}}^{\theta_\star}\text{-a.s.}$$

Note that, under $\tilde{\mathbb{P}}^{\theta_\star}$, we have that

- ▷ $y \mapsto \mathrm{p}^{\theta}(y \mid Y_{-\infty:0}) := g^{\theta}\left(\psi^{\theta}\langle Y_{-\infty:0}\rangle; y\right)$ is a density w.r.t. ν
- ▷ $y \mapsto \mathrm{p}^{\theta_\star}(y \mid Y_{-\infty:0})$ is the conditional density of Y_1 given $Y_{-\infty:0}$.

Moreover, with some continuity conditions, for n large,

$$\mathrm{L}^{\theta}_{z^{(i)},n} \approx \sum_{k=0}^{n} \ln \mathrm{p}^{\theta}(Y_k \mid Y_{-\infty:(k-1)}) \qquad \tilde{\mathbb{P}}^{\theta_\star}\text{-a.s.} \tag{10}$$

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- Taylor expansion of the score function

$$\mathbf{S}_n(\theta) = \sum_{k=0}^{n} \partial_\theta \ln \mathrm{p}^\theta(Y_k \mid Y_{-\infty:(k-1)})$$

around $\theta_\star$;

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- Prove consistency: $\lim_{n\to\infty} \widehat{\theta}_{z^{(i)},n} = \theta_\star$, $\tilde{\mathbb{P}}^{\theta_\star}$-a.s.,
- Taylor expansion of the score function

$$\mathbf{S}_n(\theta) = \sum_{k=0}^{n} \partial_\theta \ln \mathrm{p}^\theta(Y_k \mid Y_{-\infty:(k-1)})$$

  around $\theta_\star$;
- CLT for the martingale $\mathbf{S}_n(\theta_\star)$ (under $\tilde{\mathbb{P}}^{\theta_\star}$),
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The steps to the CLT for the MLE

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All theses steps can be carried out for the previously mentioned models, with some restrictions on the parameter set Θ, sometimes appearing in technical conditions.

Consistency: missing steps

From the approximation in (10), we only get that

$$\lim_{n\to\infty} \Delta\left(\widehat{\theta}_{z^{(i)},n}, \Theta^\star\right) = 0 \qquad \tilde{\mathbb{P}}^{\theta_\star}\text{-a.s.}\ , \tag{11}$$

where Δ is the metric on Θ and the limit maximizing set $\Theta^\star$ is defined for all $\theta_\star$ by

$$\Theta^\star = \operatorname{argmax}\left\{\tilde{\mathbb{E}}^{\theta_\star}[\ln p^\theta(Y_1 \mid Y_{-\infty:0})] \ :\ \theta \in \Theta\right\}$$

To prove consistency, it remains to go through two additional steps:

- we need to show that (see Douc, Roueff, and Sim [2016] for any POMM)

$$\Theta^\star = [\theta_\star] := \left\{\theta \in \Theta \ :\ \tilde{\mathbb{P}}^\theta = \tilde{\mathbb{P}}^{\theta_\star}\right\}\ .$$

  Then, with (11), we get equivalence class consistency (as introduced by Leroux [1992]).
- To conclude, find conditions to have that $[\theta_\star] = \{\theta_\star\}$. (So that the model restricted to these $\theta_\star$s is identifiable).

Identifiability: basic assumption on the observation kernel

We suppose that the observation kernel satisfies:

(A-2) We can write $\theta_\star = (\vartheta_\star, \varphi_\star)$ and, for all $\theta = (\vartheta, \varphi)$ in Θ and $x, x' \in \mathsf{X}$,

$$G^{\theta}(x; \cdot) = G^{\theta_\star}(x'; \cdot) \quad \text{if and only if} \quad \varphi = \varphi_\star \quad \text{and} \quad x = x' \; .$$

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Denote by $[\theta_\star]$ the equivalence class:

$$[\theta_\star] = \left\{ \theta \in \Theta \, : \, \tilde{\mathbb{P}}^{\theta} = \tilde{\mathbb{P}}^{\theta_\star} \right\}$$

Identifiability: a general result

We have the following result.

Theorem

Consider an ergodic ODM(p, q) satisfying (A-2). Suppose that, for all $\theta \in \Theta$, there exists $\psi^{\theta}\langle\cdot\rangle : \mathsf{Y}^{\mathbb{Z}_-} \to \mathsf{X}$ such that

$$X_1 = \psi^{\theta}\langle Y_{-\infty:0}\rangle \qquad \mathbb{P}^{\theta}\text{-a.s.} \tag{12}$$

Then $[\theta_\star]$ coincides with the set of all $\theta = (\vartheta, \varphi_\star) \in \Theta$ such that

$$\psi^{\theta}\langle Y_{-\infty:0}\rangle = \psi^{\theta_\star}\langle Y_{-\infty:0}\rangle \qquad \tilde{\mathbb{P}}^{\theta_\star}\text{-a.s. },$$

$$\psi^{\theta}\langle Y_{-\infty:0}\rangle = \psi^{\theta}_{Y_{(-p+1):0}}\left(\left(\psi^{\theta}\langle Y_{-\infty:j}\rangle\right)_{-q\leqslant j\leqslant -1}\right) \qquad \tilde{\mathbb{P}}^{\theta_\star}\text{-a.s.}$$

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Recall that (12) follows from (A-1)+(M), see ▸ likelihood approx.

Uniform Lipschitz assumption: the linear case.

For a linear link function (3) with $\upsilon(y) = y$,
Ass. (A-1) is equivalent to

(L-1) For all $\theta = (\vartheta, \varphi) \in \Theta$ with $\vartheta = (\omega, a_{1:p}, b_{1:q})$, we have $b_{1:q} \in \mathcal{S}_q$,

where

$$\mathcal{S}_q := \left\{ c_{1:q} \in \mathbb{R}^q \,:\, \forall z \in \mathbb{C}, |z| \leqslant 1 \text{ implies } 1 - \sum_{k=1}^{q} c_k z^k \neq 0 \right\} .$$

Identifiability: the linear case.

We have the following result.

Theorem

Consider an ergodic ODM(p, q) satisfying (L-1) and (A-2)+(M). Then, for any $\theta_\star = (\omega^\star, a^\star{}_{1:p}, b^\star{}_{1:q}, \varphi^\star)$ in the interior of Θ,

$$[\theta_\star] = \{\theta_\star\} \qquad \text{if and only if}$$

(L-2) The polynomials $\mathrm{P}_p(\cdot; a^\star{}_{1:p})$ and $\mathrm{Q}_q(\cdot; b^\star{}_{1:q})$ have no common complex roots,

where we defined

$$\mathrm{P}_p(z; a_{1:p}) = \sum_{k=0}^{p-1} a_{k+1}\ z^{p-1-k}$$

$$\mathrm{Q}_q(z; b_{1:q}) = z^q - \sum_{k=1}^{q} b_k z^{q-k}\ .$$

Concluding remarks

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- The same identifiability condition is valid in the general case of LODMs.
- Open question : GARCH(p, q) processes are known to be regularly varying. How can this be extended to integer valued cases ?

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