CONCEPT STUDY - PROFOUND UNDERSTANDING OF TEACHERS' MATHEMATICS

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This workshop was motivated by both current developments in the learning sciences (in particular embodied cognition) and recent shifts of how mathematics knowledge for teachers is conceptualized. We perceived a need for research mathematicians to join with grade-school mathematics educators, working together to interrogate, elaborate, and format the usually tacit associations that are used to give shape to mathematical concepts from elementary to high school. At the same time, we intended to draw deliberately on experts from different language backgrounds, endeavouring to better understand how networks of association within particular languages (including English, Spanish, Blackfoot, Mixe, and Zapoteco) might help or frustrate meaning-making of mathematical concepts. The main goal of this workshop was to gather mathematicians, mathematics educators and teachers to engage in deep cultural and linguistic analysis of mathematical concepts, and the implications for education.

A distinctive characteristic of this workshop was the commingling between two languages: Spanish and English. While mathematicians and mathematics educators came from all North America, teachers were mostly from Oaxaca and other parts of Mexico. A professional translation service was provided during the plenary presentations and conversations, and this format enabled direct interaction among indigenous teachers and teacher educators from Mexico with scholars and mathematicians from the US and Canada.

1 Overview of the Field

The relationship between mathematics teachers' disciplinary knowledge and their effectiveness in supporting learners' understandings has been a topic of intense interest among mathematics education researchers for more than four decades. As might be expected, studies have tended to focus on formal mathematics knowledge, but developments in neuroscience, complexity research and the social sciences have introduced new considerations that may have profound implications for teacher preparation and classroom practice. These implications are not yet well understood, particularly as they relate to teachers' disciplinary knowledge of mathematics. In this section we present an overview of (1) the research on mathematics teachers' disciplinary knowledge and (2) the insights that embodied cognition has provided regarding the human nature of mathematical activity, and its implications for learning.

Studies in the 1970s focused on the relationship between teachers' college credits in mathematics and the performance of their students on standardized tests, finding little or no correlation ([1], [2]). That sort of result was consistently reported well into the 1990s (cf. [3]), when the questions orienting the research began to change. Prompted by Shulman's [4] construct of "pedagogical content knowledge," a number of researchers began to delve into the particular sorts of insights that were needed to translate the tight formulations of mathematics into formats that are more accessible to young learners. By 2010, the prevailing sensibility had changed considerably. As Baumert and colleagues [5] summarized in a comprehensive review of empirical research, “findings show that [teachers’ content knowledge of mathematics] remains inert in the classroom unless accompanied by a rich repertoire of mathematical knowledge and skills relating directly to the curriculum, instruction, and student learning” (p.139). In essence, then, the principal question orienting research into teachers’ mathematics had shifted from “What formal mathematics must teachers master?” toward “What specialized mathematics do teachers require?” In the process, research methodologies shifted from focusing on teachers’ college credits and students’ examination score toward more nuanced interrogations of the structures of mathematical concepts, teachers’ capacities to unpack abstract ideas, and their abilities to enfold their insights into learning experiences for their students. For the interested reader, more fulsome accounts of this research are presented elsewhere, e.g. [5], [6], [7], [8], [9].

A more recent approach among researchers has been to pay more attention to the necessarily situated nature of teachers’ disciplinary knowledge. Ball, Thames, and Phelps [10], for example, pointed out that the mathematical knowledge of teachers is not static and argued it should be thought of as knowledge-in-action. They called for a practice-based theory of teachers’ mathematics knowledge, framed by the question, “What mathematical knowledge *is entailed in/by* the work of teaching mathematics?” This reframing was significant in that it signaled a shift in focus from knowing more mathematics to knowing mathematics differently. The shift is clearly illustrated by Ball and Bass’s ([11], [12]) research, which focused attention on a key process of teachers’ mathematical practice that they, following Ma ([13]), called *unpacking*. Unpacking is the prying apart and explicating of mathematical ideas to make sense of their constituent images, analogies, and metaphors. Taking up that emphasis, Davis ([14], [15]) has worked with teachers to develop a suite of strategies that enable collectives of educators to unpack and reformat mathematical understandings in manners more appropriate and more effective for their classrooms. Davis and Renert ([15]) elaborated on the distributed and emergent nature of the mathematics knowledge for teachers – an approach consistent with several scholars in the cognitive sciences (cf. [16]). This perspective considers knowledge as embodied in both the biological body and the social-cultural world.

Embodied cognition has received increasing attention within the learning sciences. This focus not only considers the biological for cognition, but also makes a strong emphasis on the interactions with the environment, including other individuals. Learning, from this perspective, is considered as a characteristic of living systems, from cells to multi-cell organism, to animals and to societies – e.g. ant colonies or human groups. Gerofsky ([17]) identified a number of theoretical domains of embodied cognition in mathematics education, including: philosophy; semiotics; cultural studies; linguistics/cognitive linguistics; computer science; cognitive neuroscience; education/curriculum and pedagogy; gesture studies; and fine and performing arts. We focused our attention on three features of knowledge relevant for mathematics and mathematics education.

First, Lakoff and Núñez ([18]) elaborated on the role of metaphors in the development of mathematics concepts. They identified four different meanings, or metaphors, common in mathematics for (real) numbers: object collection, object construction, using a measuring stick, and moving along a path. Each metaphor has different implications for arithmetic that, in turn, can be sources of misconceptions or limited understanding of mathematics. They went on to elaborate how set theory is formally defined and referred to using the ’container metaphor,’ and then extended to other branches of mathematics. This metaphor is based on the way we live and experience our universe as humans in a space with specific topological properties – where ’inside’ and ’outside’ are well defined. The nature of mathematics learning is, therefore, principally related to linguistic and cultural associations, which moves well beyond the symbolic realm to metaphorical associations, gestures and other forms of representations.

Second, recent developments in research into complex systems have informed a conceptualization of knowledge – including mathematics knowledge ([19]) and knowledge of mathematics for teaching ([15]). Similar to other complex phenomena, knowledge has been recast in terms of inherent complicatedness or intricateness; it is no regarded as deterministic; mathematical models of knowledge domains (including of itself) are complex and involve nonlinear, ill-posed, or chaotic behaviour; and, like all complex systems, knowledge domains such as mathematics are predisposed to unexpected outcomes. For instance, in [19] Foote presented a “test case” analyzing the development of (finite) group theory. He described a historical development of this theory identifying the properties listed in this paragraph. His description depicts mathematics knowledge as a networked, evolving structure with emergent gaps of complexity. Davis and Renert also refer to the nested structure of complex systems, in which the knowledge developed in the classroom has similarities to the broader systems of mathematical knowledge [15].

Finally, knowledge domains such as mathematics are now recognized to be distributed across professional communities, rather than residing in disciplinary silos ([20]). Professional communities of practice, such as groups of teachers or mathematicians from diverse institutes, share and produce knowledge. Although this knowledge may be "reified" in documents and tools created by the community, it develops through a common understanding, negotiated and shared by members of the community – thus, it is not "located" on individuals or texts.

2 Recent Developments and Open Problems

There have been two divergent schools of thought regarding mathematics knowledge for teaching. One holds that such knowledge is mostly formal and explicit – and so specifiable, cataloguable, unpackable, teachable, and testable (e.g. [6], [13]). The other is that such knowledge is principally tacit, and so not readily accessed or assessed through formal means (e.g., [14]). These perspectives on teacher knowledge, clearly compatible and complementary, are perhaps better represented in terms of relative emphasis than clean distinctions. However, they entail different approaches for research and development. The first is strongly focused on the individual and is associated with disseminating, testing, and measuring of both teacher and student knowledge. The second involves subtler, participatory, and time-extensive strategies to define, develop, and assess mathematics knowledge for teaching ([15]).

As yet, neither emphasis is associated with the sort of evidence base that is needed to make strong claims about teacher education or classroom practice. However, the latter has been gaining momentum in the last few years as it has been bolstered by realizations from cognitive science of the principally associative (i.e., tacit and analogical, rather than explicit and logical) nature of mathematics learning (cf., [18]), from complexity research of the networked and evolving structures of mathematics knowledge (cf., [19]), and from social sciences research of the manner in which professional knowledge is distributed across communities rather than being located in individuals or texts (cf., [20]).

Oriented by these developments, one research-and-development approach that has arisen is *concept study*, which combines elements from concept analysis and lesson study, two prominent foci in contemporary mathematics education research. Concept analysis, which was well represented in mathematics education research from the 1960s to the 1980s, focuses on explicating logical structures and figurative associations that inhere in mathematical concepts ([21]). Lesson study is a practitioner-research model that involves teachers in collaborative, small-group, longer-term inquiries to improve pedagogy ([22]).

Combining these two emphases, then, concept study refers to a collective learning structure through which teachers identify, interpret, interrogate, invent, and elaborate images, metaphors, analogies, examples, exemplars, exercises, gestures, and applications that they invoke – sometimes explicitly and sometimes implicitly – in efforts to support students' mathematical understandings. The combination of these two well-structured and rigorously researched methods already has been demonstrated both to contribute to the development of pedagogical content knowledge in mathematics and to have immediate and substantial impacts on classroom practice ([15]). Yet, some questions about its implementation in larger context, including diverse cultures, still remain to be explored.

Concept studies have helped to highlight the complexity of even "simple" or "basic" concepts, such as number, equality, and binary operations. Perhaps most significantly, concept studies can prompt teachers to a heightened awareness of some key aspects of their uses of the instantiations (e.g., images, metaphors, analogies) to frame concepts – for instance,

- how particular instantiations might enable or constrain further learning,
- how instantiations can interfere with one another in highly unproductive ways,
- how singular instantiations might be blended into more powerful ones, and
- how mathematical instantiations can operate differently in different languages (and, simultaneously, how access to diverse languages can amplify each of the above points).

Deepened understandings on these aspects can powerfully enable the work of teachers. Informed by their initial explorations of a mathematical concept, later components of concept study involve analysis of curriculum and design of mathematical tasks ([15]). Subsequently, these tasks are enacted in classrooms, followed by collective debriefing of the results to inform teachers' reflections and task refinements. One of the issues that has arisen across concept studies is that teachers' capacities to deconstruct and reconstruct concepts are limited by their horizons of mathematical knowledge. To elaborate, a concept-study group typically comprises teachers from across school grade levels. So constituted, there is an abundance of shared knowledge as to which early-grades instantiations of a concept might be productive or counterproductive in higher grades. For instance, multiplication-as-grouping is much less useful than multiplication-as-area-making or multiplication-as-scaling when moving from elementary to secondary mathematics. However, in typical concept-study groups, teachers of the higher grades do not have access to a similar source of mathematical expertise – and so often lack the knowledge needed to select and craft instantiations that will enable mathematics learning in post-secondary settings. A next step in concept-study research is thus to extend the "grade range" of participants from K – 12 to K – 20.

3 Presentation Highlights

There were four keynote presentations in the workshop, two given in English and two in Spanish. The first day of the workshop started with two presentations that introduced a conceptual framework and research results for the work undertaken during the workshop. Each presentation was followed by a session of questions and further discussion.

3.1 Day one

The first keynote speaker was Brent Davis with the presentation entitled: *Concept Study*. This presentation had two foci. On the one hand, it offered an overview of research into teachers' disciplinary knowledge of mathematics, highlighting a critical divergence between investigations aimed at cataloguing and measuring explicit knowledge and studies concerned with interrogating and reformulating implicit knowledge. On the other hand, extending the discussion of the latter concern, it presented an overview of "concept study" – an integrated-but-evolving set of strategies to uncover, analyze, and re-synthesize implicit knowledge in ways that make it more available for teaching, providing evidence to illustrate and underscore the value of such strategies for supporting effective teaching.

Davis elaborated on four critical points regarding the nature of teaching and learning. First, he explained that thinking is mainly analogical. Using examples from the classroom, he showed how students at Grade 3 described even numbers. This variety of descriptions, or meanings, was used to argue that deep meaning (and/or resilient misconception) emerges as the learner creates (or fails to create) coherence among instantiations. This meaning-making process entails more an analogical process (constantly updating a network of associations) than a logical one (deducing notions from prior notions). Second, learning is a process of construing meaning. Whereas progress in mathematics research involves the honing of definitions, mathematics learning proceeds mainly through the construing of meanings. Bruner [23] described three types of meanings. The *enactive* type is based on physical and dynamic engagement in the world. The *iconic* type of meaning is image-based and involves figures or graphics. The *symbolic* type of meaning entails the understanding and manipulation of abstract symbols. Davis described Pirie-Kieren's ([24]) taxonomy in which the symbolic, iconic, and enactive may not be considered as mutually exclusive, but rather nested. Third, the sequence of meaning-making matters for learning mathematics. Davis presented 'exponentiation' as an example of a mathematical concept in the curriculum that is addressed mainly from its symbolic meaning, despite the research evidence suggesting that deep learning is facilitated when it proceeds from the enactive to the iconinc, and then to he symbolic. He explained that teaching mathematics entails a particular type of expertise: being an expert who can think as a novice. He depicted the pedagogical content knowledge required to teach mathematics as vast, intricate, and evolving. Finally, the presentation described "concept study" in five iterative, nested levels: meanings, landscapes, analogical entailments, conceptual blending, and pedagogical problem solving. The first level, meanings, entails analyzing the variety of meanings that might be attached to a mathematical term, including meanings from common language that might impede a learner to understand a mathematical concept. Landscapes were presented as the interrelated collections of meanings associated to a mathematical concept within and beyond the curriculum. The analogical entailments level refers to the implications of considering a particular meaning in a particular context. Conceptual blending is the juxtaposition of two mathematical concepts to create a new mathematical entity. Finally, pedagogical problem solving refers to planning classroom activities to address problematic and persistent issues in teaching and learning mathematics. These five levels served to orient the work of the workshop.

The second keynote speaker was Rafael Núñez, who addressed the question *Why is embodied cognition relevant for mathematics education?* He presented mathematics as a unique body of knowledge explaining that the very entities that constitute what mathematics is are idealized mental abstractions that cannot be perceived directly through the senses. A Euclidean point, for instance, is dimensionless and cannot be empirically observed. Mathematics thus appears to be de facto "dis-embodied." Núñez challenged this position by analyzing the concept of embodied cognition and investigating why it is relevant for mathematics education. He argued that mathematics is largely realized via specific concoctions of everyday cognitive mechanisms of human sense making and abstraction. He further asserted that, although these cognitive mechanisms are natural and ordinary, the (mathematical) concoctions often are not; their learning usually require considerable cultural and educational scaffolding. Data from studies on the concept of "continuity" in calculus as well as from the investigation of one of its underlying building blocks – the "number-line"– were discussed.

In his presentation, Núñez also discussed the difficulties of translating the phrase "embodied cognition" into Spanish. The main problem was that a verbatim translation ("cognición encarnada" in Spanish) suggests that knowledge, or concepts, exists independent of human beings and learning may be understood as a process of 'inserting' this knowledge into the body. This seems to be the case of mathematics as it is presented as an abstract field, highly symbolic and in a dis-embodied language. Núñez argued, however, that mathematics is based on embodied metaphorical though – such as "sets are containers." He explained that mathematics is about 'bodily experience' mediated by embodied mechanisms for human imagination and symbolic cognition guided by cultural practices. As an example, he explained conceptions of time, from the Aymara and the Yupno communities in South America, that entail radically different conceptual outcomes.

Núñez elaborated on the field of cognitive linguistics and its methodologies to explain how different embodied conceptual mappings make human imagination possible. He explained some of these mappings – conceptual metaphor, conceptual metonymy, and fictive motion – presenting evidence from a variety of mathematical registers, including communications, definitions, and explanations.

3.2 Day two

On the second day, keynote presentations focused on indigenous mathematics education. Alicia Ávila's presentation, *Teaching mathematics and original language in indigenous schools from Mexico*, began with an overview of the conditions of mathematics education in indigenous schools in three Mexican states with relatively high populations of indigenous peoples. Ávila then explored teachers' beliefs on indigenous language and the entailments of those beliefs for mathematics teaching and learning. Finally, some cases of the use of language in classrooms with children at different fluency level in Spanish were presented.

In her presentation, Ávila addressed some challenges of indigenous education in Mexico. Since the beginning of this century, aboriginal primary schools have adopted an intercultural bilingual approach. However, the effectiveness of this approach is determined by teachers' management skills of language, and most of the teachers – although indigenous – do not speak the mother language of their students. Additionally, due to the economical opportunities afforded by knowing a mainstream language, many parents want the school to be a place of teaching Spanish or even English.

Teachers' lack of knowledge of local language represents an important pedagogical challenge in monolingual communities. Students may not understand instruction, or relay on peers' translations from Spanish to their language. The teacher may have trouble understanding questions or explanations from her students. This challenge is magnified by the fact that several mathematical terms have no direct translation into indigenous languages. Additionally, cultural practices – such as forms of spatial location – are not considered in the curriculum. These forms of location are very often rooted in the local geography and differ from the Cartesian approach in the official curriculum.

Veselin Jungic was the last keynote speaker with the presentation entitled: *The Math Catcher Outreach Program: Aims and methods*. He described the Math Catcher program, a science outreach initiative at Simon

Fraser University, British Columbia, Canada, with the objective of promoting mathematics among elementary and high school students, focusing on members of Aboriginal communities both in urban settings and on reserves. Math Catcher introduces mathematics and science to Aboriginal students through the use of First Nations imagery and storytelling. Match Catcher has produced animated films in several First Nations languages as well as bilingual picture books. In his presentation, he also gave a short history of residential schools in Canada and a brief summary of the current state of mathematics education among aboriginal groups in British Columbia.

Jungic began his presentation with some facts about aboriginal population in Canada, in particular in British Columbia. 4.3 % of the total Canadian population is First Nation, Metis or Inuit. In British Columbia, the percentage of aboriginal people is 5.4%. This population is highly diverse in culture and language.

Aboriginal peoples have a recent history of discrimination and abuse. The wounds inflicted by the colonization are still sensitive; racism is alive and stereotypes are common. The last residential school, where children were taken from their families to "kill the Indian in the child," was closed in 1996. The aboriginal population in Canada are growing faster than the non-aboriginal population, and it is estimated that it will have a significant impact on the workforce within the next decade. Jungic concluded that the problem of educating this coming wave of young aboriginal people is a major concern for both the aboriginal community and Canadian/British Columbian society as a whole.

Due to the low enrolment of aboriginal students in universities (in particular in Simon Fraser University) and the widespread perception that mathematics is one of the stumbling blocks for aboriginal students, a group of mathematicians from British Columbia created the Math Catcher Outreach Project ([25]). Its goal is to promote mathematics among elementary and high school students, as well as members of the aboriginal communities, both in urban settings and on reserves. This project was initiated in the workshop First Nations Math Education (2009) in the facilities of BIRS in Banff, Alberta. The project includes school visits, math camps, animated films, bilingual picture books, aboriginal students in math and science workshops, and academic summer camps. One of the animation was projected in the presentation, which has been translated into several languages: Blackfoot, Cree, Squamish, Heiltsuk, Halq'eme'ylem, Hul'q'umi'num', Sliammon, Nisga'a, and Huu-ay-aht.

4 Concept Study Teams

The workshop included keynote presentations and plenary conversations, as well as team-based concept studies. The three themes for the concept studies, selected collectively through an iterative process of identifying individual interests and then homing in on a narrower range of topics: number systems/numeric representation, geometry, and equality. Three large teams formed around these topics for the balance of the workshop, occasionally subdividing and regrouping and more fine-grained questions arose. A daily responsibility for each team was the preparation of a short presentation to the whole group, highlighting insights, challenges and strategies.

We sought to have a significant representation of teachers in each team. However, this was challenging as teachers initially tended to cluster together in one group. After deliberating on the need of including teachers in each group, some teachers agreed to be move to a different group. Notably one concern voiced by multiple teachers was that the specialized language that many mathematicians used in the conversations was often unfamiliar and unnecessary. This comment was shared in plenary with the whole workshop, stressing the difficulties in communication when advanced mathematical terms are used in interactions with teachers at the elementary level.

The original agenda included sessions for planning activities and tasks to be implemented in the classroom. However, the analysis of the concepts was intense and extensive and we, as a group, did not engage in planning – only a few participants reported that they engaged in planning mathematical tasks.

We presents a brief description, with some examples, of the work in each team. This report is not exhaustive of all the topics and issues addressed during the workshop. A more detailed description of this work will be published in a proceedings.

4.1 Number systems/numeric representation

This team focused on numbers systems and numeric representation worked mostly as a single large group during the first three sessions, before splitting in three subgroups. On the first day, the team began by brainstorming different meanings related to the notion of number systems. Formal representations of numbers were mentioned such as those introduced by Von Newman and Leibnitz. Representations of numbers in bases 2, 10, 20 and 60 were discussed, including their applications. The advantages of the Indo-Arabic system over the Roman system were also identified. Cultural and linguistic forms of representation of number and quantity were discussed too. Various indigenous counting systems were mentioned, including those from cultures such as Mixe, Mayan, Zapoteca, and Náuatl. Practices for measuring at some communities in Mexico were identified, including for example the use of "litre" not only for liquids, but also for grains and vegetables. Other measures related to the litre included "cuartillo" (3 litres) and "alumn" (12 litres). Also discussed was the common usage of body parts to measure length. For instance, the "cuarta" refers to the span of the fingers in one hand and is used for measuring distance. A metre is very often measured with the span of the arm. Implications for the potential learning obstacles when the formal definition of litre and metre are introduced at school were discussed.

The team also addressed differences between the way numbers are expressed in English and Spanish. For instance, when counting money in English, it is common to count by hundreds – 12 hundreds – which is not the case in Spanish. Similarly, when dates are expressed in English, years are typically broken into two two-digit numbers. For instance, 1984 is read as "nineteen, eighty-four." In contrast, in Spanish it would be read as "mil novecientos ochenta y cuatro" – that is, "one thousand, nine hundred, eighty-four."

The team analyzed the mathematical content at different school levels related to numeric systems. For Grades K to 2, the related concepts or mathematical activities included digit, counting, grouping, and mental math. For Grades 3 to 6, the team identified different forms of representation, including Roman numerals, Mayan system, base 60 (for measuring and expressing time), base 10 (including expanded notation $341 = 3 \times 10^2 + 4 \times 10 + 1$), and fractions. In junior high school students have to deal with unit conversions, equivalence between fractions and decimal representation, and irrational numbers. At senior high school students encounter number representations including: rational numbers, irrational numbers, complex numbers, binary system, positional value, and base 60 for angles.

Key features related to number, and measurement, representation were addressed. For instance, the notion of grouping and ungrouping was identified as critical for understanding different forms of representations and to represent same numbers in different systems. The team discussed that the notion of "conversion" may entail potential misunderstandings due to its meaning outside mathematics. For instance, when students are asked to "convert x Celsius degrees into Fahrenheit degrees," it is usually expected that they use a formula. However, there is nothing to convert: the temperature is the same, just measured with a different unit. The team discussed potential misconception that may be promoted by both using the word "convert" for finding equivalences and insisting to use a formula when the equivalence can be found in charts or on thermometers.

4.2 Geometry

This team focused on the concept of isometry, analyzing its relationship with the programs of studies from elementary to university levels. The following questions served to initiate the discussion.

- What do we need from basic levels for the students to comprehend the concept of isometry?
- How might teachers work on this concept with students at undergraduate and graduate levels?

The team considered different forms of approaching the concept of isometry, such as: reference points, basic transformations, and distance. Although it is possible to consider different definitions of distance, the team chose to focus on Euclidean geometry. Concepts related to isometry were identified at different levels of school and university. At the elementary level (Grades 3 to 5), the following concepts were identified: measurement, symmetry, translation, and rotation. At junior high school (Grades 6 to 9), the identified concepts were: scale, symmetry, measurement, angles, and perimeter. At senior high school the identified concepts were: scale, symmetry, proportionality, volume, and area. For the early undergraduate, or college, level (Grades 13 to 15), the team identified the concepts: translation, rotation, dilation, similarity, proportion, and congruence. Finally, for the higher levels at undergraduate, the related concepts included: translation, linear transformation, rotation, dilatation, invariance, topology, and deformation.

The team also concluded that environment and context may have an impact on the baggage that students bring to the classroom. In an urban context, Euclidian geometry may be more familiar to students, whereas other communities may locate and navigate according to the local geography – with references to, for instance, the curves that rivers form, or to circles.

4.3 Equal sign

The team that focused on the equal sign identified that "equality" may be understood as a command to perform an operation. For instance, adding $2 + 3$ for elementary students might mean staking objects. In this sense, $2 + 3$ may not be seen equal to $4 + 1$. It is common for calculus teachers to identify similar mistakes from their students' work, such as: $y = x^3 = y' = 3x^2 = y'' = 6x$.

The concept of equality may entail cultural values that can result in unexpected answers to the teacher. For instance, the words used by the Ojibwe tribe – in south-central Canada and north-central United States – for numbers not only convey cardinality, but also qualities of the objects being counted. That is, additional information about the objects in the collection is considered "culturally inseparable" from the count. Hence the number word for a collection of objects varies depending on the nature of the objects. Elders consider that to much is lost from their culture when schools focus only on the richness of the Western number system.

The team sought to identify the progression of understandings of the concept of equality at different grade levels, the progression of means of transmitting that understanding, including tight connections to proportionality, and how such an understanding can inform teaching.

At the early elementary level, students may be asked to compare collections of objects with the same number of items. However, in a manner reminiscent of the traditions of the Ojibwe tribe, young learners may pay attention to the elements in the collection. For example, a collection of three rocks is not the same as a collection of three snakes. Later, when students encounter with fractions, there may have trouble understanding the equal sign when explaining that a half of a cake is the same as two quarters: the actual objects are different. Similarly, one 10-peso coin looks very different than two 5-peso coins. For geometry, the team asked themselves the following questions.

- When do two quadrilaterals are equal?
- What does it mean to say that two triangles are equal?
- What geometric attributes define this equality?
- How and when does the notion of "equality" is introduced at school?
- What is the difference between "equality" and "equal," as in "the results is equal to"?
- What is entailed in keeping "equality" in both sides of the equality?

When approaching these questions, the team deduced that there are multiple facets of "equality" involving different mathematical objects.

The team identified different issues related to the equal sign at the secondary level, including: ratios, such as velocity; cardinality; proportion; algebraic identities; procedures; definitions; approximation; and relations.

Aspects of equality beyond mathematics were also discussed. For instance, at home students may hear from their family that "everyone is equal at this house," meaning that everyone has the same rights and obligations. However, very often it is not clear how or in what sense "everyone is equal."

5 Outcome of the Meeting

The outcomes of the workshop included a number of connections for future collaboration among participants from different institutions. In conversation with some of the participants, we identified several intentions for collaboration that emerged during the workshop, listed in the following paragraphs. While this is not an exhaustive list, it reflects the interests of educators and mathematicians for collaboration on the improvement of mathematics instruction.

Estela Navarro, an instructor from the Universidad Pedagógica Nacional, indicated an interest in working with Rocío Uicab, from the Universidad Autónoma de Ycatán, on inferential logic rules in indigenous languages – Mayan and Mixe. Estela was also invited by Homero Enriquez to present in sessions for professional learning for teachers in Oaxaca.

Quitzeh Morales, Ramiro Carrillo and Erik Díaz – three young Mexican mathematicians involved in a "Cátedra" project from the Consejo Nacional de Ciencia y Tecnología (Mexico) devoted to improve education quality in Oaxaca – had been trying to contact teachers interested in opportunities for professional development regarding mathematics education. During the workshop they had the opportunity to contact teachers from Oaxaca interested in working with mathematicians for their professional learning.

Ana Sacristán, from the Centro de Investigación y Estudios Avanzados (CINVESTAV) of the Instituto Polítecnico Nacional, and Enrique Hernández, from the Instituto de Educacin Media Superior (Mexico City), indicated their intentions for collaboration in research on mathematics education. Similarly, Enrique Hernández and Vicente Carrión, from CINVESTAV, mentioned intentions for future collaboration.

Paulino Preciado, from the University of Calgary, Enrique Hernández, and Homero Enriquez, a teacher from Red Leo in Oaxaca, discussed future engagement in teacher professional development for the summer 2016.

Xaab Nop, from the Wejën Kajën organization that focuses on mathematics education and indigenous cultures, contacted Rocío Uicab and Pilar Rosado from the Universidad Autónoma de Yucatán, regarding collaborative work on mathematics and idigenous language and culture.

A group of scholars from the Universidad Pedagógica Nacional (UPN) and the University of Calgary, included the organizers of this workshop, identified a need for a more focused work on indigenous mathematics that would consider not only linguistic and cultural references, but also geographical associations to language. This was a result of some of the conversations during the workshop. Linguistic references from many cultures are based on geographic features that directly impact the way people navigate and locate themselves, and that indirectly (e.g., through metaphor) affect how they might think about time, history, causality, relationships, and so on. In turn, such habits and associations are likely to have consequences – both enabling and constraining – for mathematical conceptualizations. Spurred by this need and inspired by the vibrant engagement of the participants in the workshop, we decided to proposed another workshop at Casa Matemática Oaxaca for 2017.

During the workshop participants from different institutions had the opportunity to advertise and invite people to pursue graduate studies in mathematics education. The invitations included the following institutions: University of Calgary, Simon Fraser University, University of British Columbia, University of Michigan, CYNVESTAV, UPN, Universidad Autónoma de Ycatán, and the Universidad Autónoma de Querétaro. Some participants indicated their interest in enrolling in the graduate programs related to mathematics education at these universities.

5.1 Comments from participants

We conclude this report with some of the remarks made by the end of the workshop. Participants were asked to explain how the workshop would impact on their learning and practice. We grouped the responses in three broader categories, described with some quotations as follows.

Impact on learning – Many people made specific references to aspects learned in the workshop, including embodied cognition as a theory of learning, concepts study as a method for deepening mathematics for teachers, and the influence of indigenous practices on mathematics teaching and learning.

> Now I have a closer perspective of works on, an theories of, mathematics education that will surely have a strong impact on the doctoral research I am initiating. The theoretical tools from Concept Study will be a perspective to be used.

> A very specific learning is that it dawned on me about Embodied Cognition. I think that now I can go back to texts I revisited time ago, but with a different eyes.

> The pursuit of more information about indigenous practices and how these influence its own cultures in terms of mathematical understandings, but also how this knowledge can play a role in mathematics teaching in other environments.

Impact on professional practice – Opinions in this theme elicited both concerns and actions regarding the teaching and learning process in the classroom and in professional development programs for teachers.

> [The workshop] generated expectations for continuing teachers professional development from an interdisciplinary perspective, because teachers required to be guided in their development process from the horizontal relationships aimed at building development projects pertinent to the realities and needs in the classroom.

> Develop activities that allow me to discover the way my students perceive the concepts, in particular I seek a way to recognize my students' conceptions. ... From what I have observed, I want to generate situations that allow my students to engage in the learning of such concepts.

Impact on networking – Several connections among participants were stressed in their written comments. Some of the opinions were about exchange of ideas and insights, as well as the opening of venues for future collaborations.

> [To] foster venues for teachers professional development in collaboration with specialists, where we teachers share experiences and approach the research knowledge in the filed of mathematics education.

> I will not fear bilingual collaboration ... In my team I was the only person who did not speak Spanish and one person spoke no English. We needed some brief time to work out how our collaboration would work, but we did work it out, and I look forward to continue working with all of these good and smart people. I urge BIRS to not abandon the model.

The comments reported by participants were consistent with the outcomes identified by the organizers, as well as the original goals of the workshop. Overall, we can conclude that the workshop met our expectations, resulting in productive venues for future collaboration among teachers, teacher educators and mathematicians.

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