Analysis on Singular Spaces

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Organizers:
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1 An overview of the workshop

This workshop centered on questions in analysis and differential geometry which arise in the presence of singularities and various forms of geometric and analytic degeneracy. The notion of singularity is very broadly construed here, both in terms of the geometric meaning and the analytical context. Incomplete Riemannian manifolds with cone or cone-edge singularities are interesting examples, as are non-compact, complete Riemannian manifolds whose asymptotic structure induces a compactification to a manifold with corners or with topological singular strata is taken to be a singular space.

The analysis of non-compact, complete Riemannian or Lorentzian manifolds via microlocal methods, which uses crucially the aforementioned compactifications, was a focus of the conference. One major question in this area is the behavior of solutions to non-elliptic differential equations. This goes back to Melrose [Me94] and his analysis of Schrodinger operators on scattering manifolds, which in particular extends Hormander's propagation of singularities theorem to "infinity", meaning up to and including the boundary of a geometrically natural compactification. This work has been extended by Vasy [V13] to treat other complete non-compact manifolds, first to asymptotically hyperbolic manifolds, but more recently by Hintz and Vasy [HV16] to treat difficult questions in quasilinear hyperbolic PDE's arising in mathemetical general relativity.

A related topic is the structure of Green's functions and the construction of natural pseudodifferential calcului, both on non-compact Riemannian manifolds, and on incomplete Riemannian manifolds with structured singularities. Structure theorems for the integral kernels of these operators yield powerful tools for treating many of the questions which arise in linear elliptic analysis, including index theory, on non-smooth or non-closed spaces. The foundational work of Atiyah-Patodi-Singer for manifolds with boundary points to the enormous richness of interaction between linear analysis and geometric degeneracy.

The conference treated these topics and many others arising in the study of singular spaces, questions ranging from foundational geometric analysis questions (see Núñes-Zimbrón's talk on harmonic functions on singular spaces) to higher Teichmüller theory and its relationship to degenerate families of solutions to non-linear PDEs.

2 Presentation Highlights

Andras Vasy discussed recent work on the low energy behaviour of the resolvent for Schrödinger operators. This is related to the inventiones paper [HHiV21] showing the linear stability of the Kerr family of spacetimes. Vasy has developed a method, building off of Melrose's formulation of radial points estimates for real principle type operators, for constructing "Fredholm problems" from differential operators which are not elliptic but whose characteristic sets have special structure. His basic construction goes back to 2011 and his work on Asymptotically Hyperbolic manifolds. In this talk he explained how one can use second microlocalization to prove a family of resolvent estimates which hold uniformly down to zero energy.

Yaiza Canzani gave a talk on joint work with Galkowski on bounds for eigenfunctions on closed manifolds, related to their work in [CG21]. She explained the interesting method they utilized of decomposing eigenfunctions in terms of geodesic beams, a process by which one expresses these functions as a superposition of functions supported in phases space on tubes associated to geodesics. They prove a pointwise $\sqrt{log\lambda}$ (with λ the frequency) improvement to the universal L^∞ bound, under an assumption on the geodesic flow which is the absence of what they call "uniform maximal self conjugacy", improving (in the sense of relaxing assumptions) on a result of Berard which assumes no conjuage points globally. She also explained a newer result in which they improve the related L^p-bounds under a global version of this same assumption, and a related improvement for Weyl laws.

Colin Guillarmou gave a beautiful overview of his work on the mathematics of Conformal Field Theory. In particular he explained his work in collaboration with Kupiainen, Rhodes, and Vargas, in which they give a rigorous derivation of the conformal bootstrap hypothesis. See [GKRV20].

Rafe Mazzeo talked on work in progess with Haydys and Takahashi on $\mathbb{Z}_2$ harmonic spinors. This work is related to higher Teichmüller theory, in particular the study of the character variety of a closed manifold with respect to $SL(2,\mathbb{C})$ (or more generally for a complexified semisimple Lie group). Representing points in this character variety as complex flat connections on a fixed rank 2 complex vector bundle, they consider the boundary features of this manifold by looking at divergent sequences of connections; such sequences can be shown upon renormalization to produce, in the limit, harmonic 1-forms valued in line bundles with $\mathbb{Z}_2$-valued monodromy with respect to a curve Γ on which the forms vanish. Analogues for different Lie groups produce related limiting objects, which are refered to more generally as $\mathbb{Z}_2$-harmonic spinors. He described ongoing work in which, subject to various assumptions, including on the regularity of the zero set of the spinors, they proved that the linearized deformation problem is equivalent to the deficiency spaces of a related elliptic boundary value problem.

Hadrian Quan, a recently completed PhD student, gave an excellent talk on his work on asymptotically complex hyperbolic manifolds, in which he extended the Fredholm problem constructions of Vasy mentioned above to asymptotically

hyperbolic spaces. In a addition to this he proved a structure theorem for the wave trace using a parametrix construction which incorporates the rescaled bundles mentioned above.

Melissa Tacy gave a talk on structures of random waves on compact manifolds. At high energies, eigenfunctions of the Laplacian on domains in the plane, appear in numerical analysis to exhibit a filiment structure; which is essentially a network of interconnected segments along which the functions concentrate. Her detailed analysis rules out an L^2 concentration along lines but allows for the possibility of a family of so-called "bright spots" which are logarithmic concentration of L^2-mass, to accumulate along rays without contradicting the equidistribution of the L^2-mass along these rays. [T21].

Fŕedéric Rochon explained his work with Chris Kottke [KR], in which they prove that on quasi-fibered boundary metric of depth 2 satsifying some assumptions, the Hodge-deRham operator induces a Fredholm operator between appropriate Sobolev spaces. In addition to this they prove that the L^2-harmonic forms have a positive rate of decay. This work combined with that of Fritsch-Kottke-Singer [FKS] yields a proof of the Sen conjecture in depth 2, and also the Vafa-Witten conjecture within some range of parameters. Their work uses a geometric microlocal parametrix construction.

Raquel Perales explained her results in [HLP] extending the classic results of Gromov and separately Cheeger-Colding which show that manifolds of dimension m which have uniformly bounded-below Ricci curvature, volume bounded uniformly below, and diameter uniformly bounded above form a Gromov-Hausdorff precompact, and their limits are $C^{1,\alpha}$ off a set of Hausdorff dimension at most $m-2$. They show that for totally convex spaces, the Ricci curvature condition can be replaced by a lower bound for the metric itself, provided the metric used to define convergence is changed to the "intrinsic flat metric". They give a similar result for sequences of complete non-compact manifolds with almost non-positive ADM mass.

Kiril Datchev gave a talk on his work with his undergraduate research advisee Nkhalo Malawo [DM], computing the resonance width asymptotics for the delta potential on the half-line, by deriving a formula for resonances in terms of the Lambert W function and applying a series expansion. This work refines results of Galkowski-Smith [GS]. The δ potential is a simple model of a thin barrier, motivated by physical problems such as quantum corrals and leaky quantum graphs.

Jesús Núñez-Zimbrón gave a talk on joint work with Guido de Philippis [PN19] in which they study harmonic functions on singular spaces which admit a generalized notion of Ricci curvature bounds. This generalization is obtained through the Bochner inequality, which is equivalent to a lower Ricci bound in the smooth case and generalizes to certain singular spaces under assumptions obtained using the theory of metric measure spaces. They prove in particular the harmonic functions on such spaces have vanishing gradient at "bad" points, where bad is defined in terms of the diameter of the tangent cone.

Semyon Dyatlov described work with Cekić, Küster, and Paternain on the vanishing order of the Ruelle zeta function on conformal metric perturbations of

compact hyperbolic 3−manifolds. The Ruelle zeta function is defined in terms of the lengths of closed geodesics, and the authors build off work of Dyatlov and Zworski which uses microlocal methods to prove that it is meromorphic on the complex plane. In this talk, Dyatlov explained that they find topological formula for the vanishing order in terms of the first Betti number.

3 Scientific Progress / Outcome of the Meeting

The workshop was held virtually, with talks on Zoom and informal discussion on Gathertown. There were talks from both junior and senior researchers, including a talk from a graduating phd student. There were active question and answer sessions after each talk, and a healthy exchange of ideas between the various represented fields. There progress made in a variety of settings. In particular the results of Rochon show that the full proof of the Sen conjecture is likely not far off. Also the exciting possibility of further application of microlocal methods to conformal field theory indicated by Guillarmou's talk was a highlight.

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