Report on the workshop
Lipschitz Geometry of Singularities
Casa Matematica, Oaxaca, 21-26 October 2018

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1. Overview of the Field

"Lipschitz geometry" refers to the category of metric spaces and Lipschitz maps; isomorphism in this category is bi-Lipschitz homeomorphism. A real analytic space germ $(V,p) \subset (\mathbb{R}^N, 0)$ has two natural Lipschitz geometries induced from the standard euclidian metric on $\mathbb{R}^N$. The outer metric is defined by the restriction of the distance in $\mathbb{R}^N$, while the inner metric is defined by the infimum of lengths of paths in V. When (V,p) is a complex analytic germ, both metrics are "natural" in the sense that in the Lipschitz category they are independent of the choice of holomorphic embedding in some $(\mathbb{C}^N, 0)$.

Lipschitz geometry of singular sets is an intensively developing subject which started in 1969 with the work of Pham and Teissier on the bi-Lipschitz classification of germs of plane complex algebraic curves. These results were revisited later from a geometrical point of view by Fernandes and Neumann-Pichon. Its study in complex dimension 2 and higher has expanded rapidly since then.

What makes Lipschitz geometry of singularities attractive is that it gives tame classifications: the set of equivalence classes of complex algebraic sets in $\mathbb{C}^N$ defined by polynomial equations of bounded degree is finite. One of the most important results in Lipschitz geometry is the proof of this statement by Mostowski using his theory of Lipschitz stratifications. An analogous real result, conjectured by Sullivan and Siebenmann, was proven by Parusiński in the semi-algebraic setting. The recent theory of Valette significantly improved the theory of Mostowski and Parusiński, extending the results to the category of definable sets in o-minimal structures.

2. Recent Developments and Open Problems

There has been considerable recent progress in Lipschitz geometry of complex algebraic surfaces. In fact, despite much activity in the real semi-algebraic context, it was believed by some experts that complex germs are simply metrically conical (i.e., bi-Lipschitz equivalent to the metric cone on their links), so inner Lipschitz geometry would say nothing more than topology. But the first counter-example was found by Birbrair and Fernandes in 2006, and then Birbrair, Fernandes and Neumann discovered several interesting geometric phenomena, such as fast loops and separating sets which soon made clear that metric conicalness is the exception rather than the rule. The subject then developed very rapidly, leading to the complete classification by Birbrair, Neumann and Pichon of the inner Lipschitz geometries of germs of normal complex surfaces, and building on it to several works on outer geometry of complex surfaces.

In particular, using outer geometry Neumann and Pichon proved the equivalence of Lipschitz equisingularity and Zariski equisingularity for germs of surfaces and that many analytic invariants such as multiplicity, topology of discriminants and polars, etc. are determined by outer geometry of surface germs. There has also been interesting work on "Lipschitz normal embedding" (LNE, = bilipschitz equivalence of outer and inner geometry). For example Fernandes and Sampaio proved necessary condition for an algebraic set to be normally embedded. Birbrair and Mendes gave a characterization of LNE algebraic germs. Then it was shown by by Neumann, Pedersen and Pichon that minimal singularities are characterized among rational surface singularities by being normally embedded, a result which is a tantalizing step towards Lê Dũng Tráng's suggested duality between resolution of surface singularities by repeated normalized blowing up and resolution by repeated normalized Nash transform.

There are some partial studies of Lipschitz geometry for families of singularities in higher dimensions, such as Birbrair, Fernandes, Gaffney and Grandjean, and other higher dimensional results were also obtained recently. For example, it was shown by Birbrair, Fernandes, Lê and Sampaio that Lipschitz regularity of complex analytic sets implies smoothness and Sampaio proved that bi-Lipschitz equivalent semialgebraic sets have bi-Lipschitz equivalent tangent cones (the topological equivalence of tangent cones was conjectured by Zariski but disproved by Bobadilla). Moreover, in a recent paper of K.U.Katz, M.G. Katz, D.Kerner and Y. Liokumovich ("Determinantal variety and normal embedding", J. Topol. Anal. 10 (2018), 27–34) and more general results of Kerner, Petersen and Ruas ("Lips chitz normal embeddings in the space of matrices" Math. Z. 290 (2018), 485–507), normal embedding is proved for various determinantal singularities. However, the general understanding of Lipschitz geometry in higher dimensions and for non isolated singularities is not yet well developed and will provide significant research projects well into the future.

An important long-standing question in singularity theory is Zariski's multiplicity conjecture (actually stated as a question), which states that the multiplicity of an algebraic hypersurface germ is determined by its embedded topological type. A Lipschitz version of Zariski multiplicity conjecture was proved by Fernandes and Sampaio for complex hypersurface germs (not necessarily with isolated singularities). The analogous result for normal surface germs (of any embedding dimension) was proved earlier Neumann and Pichon. The higher dimensional case remains open, as well as the non-Lipschitz version in all generality, and is being very actively investigated.

There are also important new results in Lipschitz geometry of germs of functions. Birbrair, Fernandes, Gabrielov and Grandjean obtained finite Lipschitz classification of germs of definable functions on the real plane (Lipschitz K-equivalence). Notice that Lipschitz A-equivalence has so-called "moduli" and does not allow discrete classification (Henry and Parusinski). In work in progress, Gaffney, Neumann, Pichon and Teissier extended the collection of Henry-Parusinski moduli for germs of functions $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and are close to a full Lipschitz A-equivalence classification of such germs. Again, higher dimensions remain a challenge.

3. Presentation Highlights

According to the pre-mentioned major scientific problems, Lipschitz geometry has potential promising applications in several different areas of singularity theory. The workshop focussed not only on the understanding of the Lipschitz geometry of singular spaces and maps but also on its interaction with other aspects of singularities. The scientific committee classified the talks in several themes: resolutions, abstract and embedded topology of hypersurfaces, equisingulity, deformations, smoothings, natural stratifications, arc and jet spaces, geodesics on singular spaces, curvature, etc. Part of the meeting concentrated on the questions and methods which emerged recently in the complex setting, but it will leave a large place for questions arising in both complex and real settings, and it will promote exchanges between complex and real algebraic geometers and topologists.

The topics covered by the meeting included:

(1) Lipschitz Geometry of germs complex surfaces. This contained some introductory lectures for students and post-docs.
(2) Zariski Multiplicity Conjecture. This included an elementary talk on regularity and invariance of tangent cones, and a review of recent results in this direction.
(3) Lipschitz geometry and resolution of singularities
(4) Equisingularity
(5) Lipschitz geometry of Determinantal manifolds
(6) Lipschitz geometry and arc spaces and jet schemes
(7) Toric geometry

4. Scientific Progress Made

The participants learned several new techniques and scientific progress to understand and Lipschitz geometry of singularities. The group discussion touched on most of the challenges and scientific issues related to topological methods and algebraic methods. Participants discussed various ways to build new invariants and new directions to explore in the future. All these discussions were very productive especially for the young scientists learning new and advance scientific techniques from the leading scientist working in Lipschitz geometry of singularities.

Among the major advances presented during the meeting, we may quote:

- **The Lipschitz Classification of definable Surface Singularities.** Birbrair and Gabrielov consider the problem of Lipschitz classification of singularities of Real Surfaces definable in a polynomially bounded o-minimal structure (e.g., semialgebraic or subanalytic) with respect to the outer metric. The problem is closely related to the problem of classification of definable functions with respect to Lipschitz Contact equivalence. Invariants of bi-Lipschitz Contact equivalence presented in Birbrair et al. (2017) are used as building blocks for the complete invariant of bi-Lipschitz equivalence of definable surface singularities with respect to the outer metric.

- **Ultrametrics and non-archimedean geometry of singularities** Fantini introduced a non-archimedean version of the link of a singularity. This object is a space of valuations, a close relative of non-archimedean analytic spaces (in the sense of Berkovich) over trivially valued fields. From the structure of these links, Ihe deduces information about the resolutions of surface singularities. Then, in collaboration with Favre and Ruggiero, he characterizes those normal surface singularities whose link satisfies a self-similarity property.

Popescu-Pampu presented common results with Garca Barroso and Gonzlez Pérez in the same direction. Favre and Jonsson described in 2004 from various viewpoints a structure of real tree on the projectified space of semivaluations centered at a smooth point of a complex surface. Popescu-pampu at al. describe it from another viewpoint, as the projective limit of the Eggers-Wall trees of the reduced germs of curves at the given point. The Eggers-Wall tree measures the contacts between the various Newton-Puiseux series of a given curve singularity. Therefore, their viewpoint is adapted whenever one studies the local geometry of surfaces using such series.

- **On multiplicity of singularities as bi-Lipschitz invariant.** Fernandes, with Fernandez de Bobadilla, Birbrair, J. E. Sampaio and M. Verbitsky proved that multiplicity of complex analytic singularities of dimension d is invariant under bi-Lipschitz homeomorphisms if, and only if, $d = 1$ or $d = 2$.

- **The smooth Whitney fibering conjecture and Whitney cellulation.** In a joint work with Murolo and du Plessis, Trotman proved the smooth Whitney fibering conjecture, in particular for every stratum X of a Whitney stratified set, locally near points of X the foliation defined by the Thom-Mather topological trivialization can be chosen, via suitable vector fields, so that the tangent spaces to the leaves are continuous at X. Moreover the associated wings have a similar property and are Whitney regular. As an application he proved with Claudio Murolothat every compact Whitney stratified set admits a Whitney cellulation, i.e. a cellulation such that the cells form a Whitney stratification. This resolves a homology problem of Goresky.

- **On Lipschitz normal embedding.** The germ of an algebraic variety is naturally equipped with two different metrics up to bilipschitz equivalence. The inner metric and the outer metric. One calls a germ of a variety Lipschitz normally embedded if the two metrics are bilipschitz equivalent. Pedersen proved Lipschitz normal embeddedness of some algebraic subsets of the space of matrices. These include the space of matrices, symmetric matrices and skew-symmetric matrices of rank equal to a given number and their closures, the upper triangular matrices with determinant 0 and linear space transverse to the rank stratification away from the origin.

  Pichon presented a characterization of Lipschitz normally embedding among normal surface singularities as a joint work woth Neumann and Pedersen. Misev presented, in a joint work with Pichon, an infinite family of Lipschitz normally embedded singularities among superisolated hypersurface singularities in $(\mathbb{C}^3, 0)$. The proof was based on the characterization of Lipschitz normal embedding presented in Pichon's talk.

- **Classification of Lipschitz simple germs.** Nguyen and Trivedi presented some bi-Lipschitz invariants that are used in their recent work on classification of Lipschitz simple germs.The invariants include rank, corank and algebraic tangent cone of non-quadratic part of function germs. Then, they presented techniques for checking bi-Lipschitz triviality of one-parameter deformations and give a list of smooth one-modal germs that are bi-Lipschitz trivial. Furthermore, they presented the complete list of Lipschitz simple germs.

- **Relations between polynomial solutions, extensions, radical ideals and Lipschitz normal embeddings** Michalska considers polynomials $f, g \in k[X]$, where k is the field of complex or real numbers. Under certain assumptions she showed equivalence of the following conditions:
  (i) (f, g) is radical
  (ii) for every polynomial h if there exists a pointwise solution of
  $$A \cdot f + B \cdot g = h$$
  then there exists its polynomial solution
  (iii) every continuous function
  $$F = \begin{cases} \alpha & on\ \{f = 0\} \\ \beta & on\ \{g = 0\} \end{cases}$$
  with $\alpha, \beta \in k[X]$, is a restriction of a polynomial.

  She then established some relations of (i-iii) with Lipschitz normal embedding.
- **Infinitesimal Lipschitz Equisingularity: Genericity and Necessity.** In earlier work, Gaffney used the double of an ideal to define in integral closure terms a notion of infinitesimal Lipschitz equisingularity for hypersurfaces. During the meeting, he described different ways of extending this notion to general spaces, and show the most restrictive version is a generic condition. He then showed that this condition is necessary for strong bi-Lipschitz equisingularity, a notion due to Fernandes and Ruas. Hence his condition gives a necessary condition for a smooth subset of an analytic set to be a stratum in a Mostowski stratification of the set.
- **Moderately discontinuous homology.** Fernandez de Bobadilla presented a joint work in progress with Heinze, Pe pereira, and Sampaio on a homology theory for singularities that is able to capture outer metric phenomena. For example it is a complete invariant for the outer metric of complex plane curves.

5. Outcome of the Meeting

This meeting brought together various mathematical backgrounds in singularity theory. The meeting was very timely and useful, and will surely have a strong impact on the further developments of Lipschitz geometry of singular spaces and maps and related subjects. The organizers were in particular pleased to observe the many young people interested in the subject; some of them have already made substantial contributions (as witnessed by the talks given by junior researchers) and will surely continue to advance the subject. The meeting provided a great overview of the field, with a number of excellent talks describing some recent exciting results in several research directions. It is quite certain that some new interesting results and collaborations will emerge as a result of the workshop.